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Positive fractional continuous-time linear systems with singular pencils

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Abstract. A method for checking the positivity and finding the solution to the positive fractional descriptor continuous-time linear systems with singular pencils is proposed. The method is based on elementary row and column operations of the fractional descriptor systems to equivalent standard systems with some algebraic constraints on state variables and inputs. Necessary and sufficient conditions for the positivity of the fractional descriptor systems are established.

Key words: positive, fractional, descriptor, continuous-time, linear, system, singular pencil.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in [1–3]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4–6]. Standard descriptor control systems have been addressed in [7, 8]. Positive descriptor linear systems have been analyzed in [9–12]. A method based on shuffle algorithm for checking of the positivity of descriptor linear systems with a regular pencil has been proposed in [12]. Singular fractional systems and electrical circuits have been addressed in [13].The stability of positive descriptor systems has been analyzed in [14].

In this paper a method for checking the positivity and finding the solution to the positive fractional descriptor continuous-time linear systems with singular pencils is proposed.

The paper is organized as follows. In Sec. 2 some definitions and theorems on fractional linear continuous-time systems are recalled and the problem formulation is given. The main result of the paper is presented in Sec. 3 where a method for checking the positivity and finding the solution to the positive fractional descriptor system with a singular pencil is presented. Concluding remarks are given in Sec. 4.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix.

2. Preliminaries and the problem formulation

In this paper the following Caputo definition of the derivative-integral of fractional order is used [15]

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(w-\alpha)} \int_0^t \frac{f^{(w)}(\tau)}{(t-\tau)^{\alpha+1-w}} d\tau, \tag{1a}$$

$$ w-1 < \alpha < w, \qquad w \in N = \{1,2,...\}, $$

where

$$ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad \mathrm{Re}(x) > 0 \tag{1b}$$

is the gamma function and

$$ f^{(w)}(\tau) = \frac{d^w f(\tau)}{d\tau^w} \tag{1c}$$

is the classical w order derivative.

Consider the continuous-time fractional linear systems described by the state equations

$$ \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \qquad 0<\alpha<1 \tag{2a}$$

$$ y(t) = Cx(t) + Du(t), \tag{2b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Theorem 1. [15] The solution of the Eq. (2a) satisfying the initial condition $x(0) = x_0$ has the form

$$ x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \tag{3}$$

where

$$ \Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{4}$$

$$ 0<\alpha<1. $$

Definition 1. [15] The fractional system (2) is called the (internally) positive fractional system if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$,

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$t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Let e_i be the i-th column (row) of the identity matrix I_n. A column (row) of the form ae_i, $a > 0$ is called monomial. A square matrix $A = [a_{ij}]$ is called Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$ [2, 15–17].

Theorem 2. [15] The fractional system (2) is (internally) positive if and only if

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \quad (5)$$

Consider the descriptor continuous-time linear system with a singular pencil

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad (6a)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{q\times n}$, $B \in \Re^{q\times m}$, rank $E \leq \min(q, n)$ and

$$\begin{gathered}\text{rank}\,[Es - A] \leq \min(q, n) \\ \text{for some } s \in C \text{ (the field of complex numbers).}\end{gathered} \quad (6b)$$

Let U_{ad} be a set of all given admissible inputs $u(t) \in \Re^m$ of the system (6a). A set of all initial conditions $x_0 \in R^n$ for which the Eq. (6a) has a solution $x(t)$ for $u(t) \in U_{ad}$ is called the set of consistent initial conditions and is denoted by X_c^0. The set X_c^0 depends on the matrices E, A, B but also on $u(t) \in U_{ad}$ [8].

Theorem 3. [8] The Eq. (6) has a solution $x(t)$ for all $u(t) \in U_{ad}$ and zero initial conditions if and only if

$$\begin{gathered}\text{rank}\,[Es - A] = \text{rank}\,[Es - A, B] \\ \text{for some } s \in C.\end{gathered} \quad (7)$$

Now let us consider the fractional descriptor continuous-time linear system with a singular pencil

$$E\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (8)$$

where $0 < \alpha < 1$ is the fractional order and $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{q\times n}$, $B \in \Re^{q\times m}$, rank $E \leq \min(q, n)$ and (6b) holds.

Definition 2. The fractional descriptor systems (8) is called (internally) positive if $x(t) \in R_+^n$, $t \geq 0$ for every consistent nonnegative initial condition $x_0 \in X_{c+}^0$ and all admissible inputs $u(t) \in U_{ad+}$, where X_{c+}^0 and U_{ad+} are the set of nonnegative consistent initial conditions and nonnegative admissible inputs, respectively.

The problem under considerations can be stated as follows. Given the fractional descriptor system (8) with a singular pencil, find conditions under which the system is positive.

The following elementary row (column) operations is used:

1. Multiplication of the i-th row (column) by a real number c. This operation is denoted by $L[i \times c]$ ($R[i \times c]$).
2. Addition to the i-th row (column) of the j-th row (column) multiplied by a real number c. This operation is denoted by $L[i + j \times c]$ ($R[i + j \times c]$).
3. Interchange of the i-th and j-th rows (columns). This operation is denoted by $L[i, j]$ ($R[i, j]$).

3. Reduction of descriptor systems to standard systems

Consider the fractional descriptor system (8) with singular pencils. In this section using elementary row operations the descriptor systems are reduced to equivalent standard systems with some algebraic constrains on state variable and inputs.

Performing elementary row operations on the array

$$E \quad A \quad B \quad (9)$$

or equivalently on the Eq. (8) we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} \quad (10)$$

and

$$E_1\frac{d^\alpha x(t)}{dt^\alpha} = A_1x(t) + B_1u(t), \quad (11a)$$

$$0 = A_2x(t) + B_2u(t) \quad (11b)$$

where $E_1 \in \Re^{r\times n}$ has full row rank. If rank $E_1 = r$ then there exists a nonsingular matrix $P \in \Re^{n\times n}$ of elementary column operations such that

$$\begin{gathered}E_1P = [\ \overline{E}_1 \quad 0\ ], \qquad \overline{E}_1 \in \Re^{r\times r}, \\ 0 \in \Re^{r\times(n-r)}, \qquad \det \overline{E}_1 \neq 0.\end{gathered} \quad (12)$$

Defining the new state vector

$$\begin{gathered}\overline{x}(t) = P^{-1}x(t) = \begin{bmatrix} \overline{x}_1(t) \\ \overline{x}_2(t) \end{bmatrix}, \\ \overline{x}_1(t) \in \Re^r, \qquad \overline{x}_2(t) \in \Re^{n-r}\end{gathered} \quad (13)$$

and using (12) we may write the equations (11) in the form

$$\begin{gathered}E_1PP^{-1}\frac{d^\alpha x(t)}{dt^\alpha} = \overline{E}_1\frac{d^\alpha \overline{x}_1(t)}{dt^\alpha} = A_1PP^{-1}x(t) \\ + B_1u(t) = \overline{A}_{11}\overline{x}_1(t) + \overline{A}_{12}\overline{x}_2(t) + B_1u(t),\end{gathered} \quad (14a)$$

$$\begin{gathered}0 = A_2PP^{-1}x(t) + B_2u(t) \\ = \overline{A}_{21}\overline{x}_1(t) + \overline{A}_{22}\overline{x}_2(t) + B_2u(t),\end{gathered} \quad (14b)$$

where

$$\begin{gathered}A_1P = [\ \overline{A}_{11} \quad \overline{A}_{12}\ ], \qquad \overline{A}_{11} \in \Re^{r\times r}, \\ \overline{A}_{12} \in \Re^{r\times(n-r)}, \qquad A_2P = [\ \overline{A}_{21} \quad \overline{A}_{22}\ ], \\ \overline{A}_{21} \in \Re^{(q-r)\times r}, \qquad \overline{A}_{22} \in \Re^{(q-r)\times(n-r)}.\end{gathered} \quad (14c)$$

Case 1.
If

$$q > n \quad \text{and} \quad \text{rank}\,\overline{A}_{22} = n - r, \quad (15)$$

then performing elementary row operations on the array

$$\overline{A}_{21} \quad \overline{A}_{22} \quad B_2 \quad (16)$$

or equivalently on (14b) we obtain

$$\begin{matrix} \overline{A}_{31} & \overline{A}_{32} & \overline{B}_2 \\ \hat{A}_{31} & 0 & \hat{B}_2 \end{matrix} \quad (17)$$

and

$$0=\overline{A}_{31}\overline{x}_1(t)+\overline{A}_{32}\overline{x}_2(t)+\overline{B}_2u(t), \tag{18a}$$

$$0=\widehat{A}_{31}\overline{x}_1(t)+\widehat{B}_2u(t), \tag{18b}$$

where

$$\begin{gathered}\overline{A}_{31}\in\Re^{(n-r)\times r}, \qquad \overline{A}_{32}\in\Re^{(n-r)\times(n-r)},\\ \overline{B}_2\in\Re^{(n-r)\times m}, \qquad \det\overline{A}_{32}\neq 0,\\ \widehat{A}_{31}\in\Re^{(q-n)\times r}, \qquad \widehat{B}_2\in\Re^{(q-n)\times m}.\end{gathered} \tag{18c}$$

From (18a) we have

$$\overline{x}_2(t)=-\overline{A}_{32}^{-1}\overline{A}_{31}\overline{x}_1(t)-\overline{A}_{32}^{-1}\overline{B}_2u(t). \tag{19}$$

Substituting of (19) into (14a) yields

$$\begin{aligned}\overline{E}_1\frac{d^\alpha\overline{x}_1(t)}{dt^\alpha}&=(\overline{A}_{11}-\overline{A}_{12}\overline{A}_{32}^{-1}\overline{A}_{31})\overline{x}_1(t)\\ &+(\overline{B}_1-\overline{A}_{12}\overline{A}_{32}^{-1}B_2)u(t)\end{aligned} \tag{20}$$

and after premultiplying of (20) by $\overline{E}_1^{-1}$ we obtain

$$\frac{d^\alpha\overline{x}_1(t)}{dt^\alpha}=\overline{A}_1\overline{x}_1(t)+\overline{B}_1u(t), \tag{21a}$$

where

$$\begin{aligned}\overline{A}_1&=\overline{E}_1^{-1}(\overline{A}_{11}-\overline{A}_{12}\overline{A}_{32}^{-1}\overline{A}_{31})\in\Re^{r\times r},\\ \overline{B}_1&=\overline{E}_1^{-1}(B_1-\overline{A}_{12}\overline{A}_{32}^{-1}\overline{B}_2)\in\Re^{r\times m}.\end{aligned} \tag{21b}$$

From (18) it follows that the consistent initial condition

$$\overline{x}_0=\begin{bmatrix}\overline{x}_{10}\\ \overline{x}_{20}\end{bmatrix}$$

and the admissible inputs $u(t)$ should satisfy the conditions

$$\begin{aligned}0&=\overline{A}_{31}\overline{x}_{10}+\overline{A}_{32}\overline{x}_{20}+\overline{B}_2u(0)\\ \text{and}\quad 0&=\widehat{A}_{31}\overline{x}_{10}+\widehat{B}_2u(0).\end{aligned} \tag{22}$$

Note that by (13) $\overline{x}_0\in R_+^n$ for any $x_0\in R_+^n$ if and only if $P^{-1}\in R_+^{n\times n}$.

Case 2.

If the condition (15) is not satisfied then from (14a) we have

$$\frac{d^\alpha\overline{x}_1(t)}{dt^\alpha}=\widehat{A}_1\overline{x}_1(t)+\widehat{A}_2\overline{x}_2(t)+\widehat{B}_1u(t), \tag{23a}$$

where

$$\widehat{A}_1=\overline{E}_1^{-1}\overline{A}_{11}, \quad \widehat{A}_2=\overline{E}_1^{-1}\overline{A}_{12}, \quad \widehat{B}_1=\overline{E}_1^{-1}B_1. \tag{23b}$$

In this case the consistent initial condition $\overline{x}_0$ and the admissible inputs $u(t)$ should satisfy the conditions

$$0=\overline{A}_{21}\overline{x}_{10}+\overline{A}_{22}\overline{x}_{20}+\overline{B}_2u(0). \tag{24}$$

Note that choosing arbitrary $\overline{x}_2(t)\in R_+^{n-r}$, $t\geq 0$ and using (3) for (23a) we can find

$$\begin{aligned}\overline{x}_1(t)&=\Phi_{10}(t)\overline{x}_{10}\\ &+\int_0^t\Phi_1(t-\tau)[\widehat{A}_2\overline{x}_2(\tau)+\widehat{B}_1u(\tau)]d\tau\end{aligned} \tag{25}$$

for given $\overline{x}_{10}$ and $u(t)$, where $\Phi_{10}(t)$ and $\Phi_1(t)$ are defined by (4) for $A=A_1$. By Theorem 2 the system (21) is positive if and only if

$$\overline{A}_1\in M_r, \qquad \overline{B}_1\in\Re_+^{r\times m} \tag{26}$$

and the system (23) is positive if and only if

$$\widehat{A}_1\in M_r, \qquad \widehat{A}_2\in\Re_+^{r\times(n-r)} \quad\text{and}\quad \widehat{B}_1\in\Re_+^{r\times m}. \tag{27}$$

Therefore, the following theorem has been proved.

Theorem 3. Let the fractional descriptor system (8) satisfy the assumptions (6b). The system satisfying the condition (15) and $P^{-1}\in\Re_+^{n\times n}$ is positive for nonnegative consistent initial condition and admissible inputs if and only if the condition (26) is met. If the condition (15) is not satisfied then the descriptor system is positive if and only if the conditions (27) are met and $\overline{x}_2(t)\in R_+^{n-r}$, $t\geq 0$ can be chosen arbitrarily so that (14b) holds.

Using the formula (3) to standard fractional system we can find a solution $x(t)$ to the descriptor system (8).

The presented method of checking of the positivity of fractional descriptor systems (21) is illustrated by the following two simple numerical examples.

Example 1. Consider the fractional descriptor system (8) with the matrices

$$E=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, \quad A=\begin{bmatrix}-2&0\\0&1\\1&-1\end{bmatrix}, \quad B=\begin{bmatrix}1\\0\\-1\end{bmatrix}. \tag{28}$$

In this case $q=3$, $n=2$, $m=1$, rank $E=1$

$$\text{rank}\,[Es-A]=\text{rank}\begin{bmatrix}s+2&0\\0&-1\\-1&1\end{bmatrix}=2 \tag{29}$$

for some $s\in C$ and

$$\begin{gathered}E_1=[\,1\quad 0\,], \qquad A_1=[\,-2\quad 0\,], \qquad B_1=[1],\\ A_2=\begin{bmatrix}0&1\\1&-1\end{bmatrix}, \qquad B_2=\begin{bmatrix}0\\-1\end{bmatrix}.\end{gathered} \tag{30}$$

The matrix

$$P=\begin{bmatrix}1&0\\0&1\end{bmatrix}, \qquad \overline{E}_1=[1], \qquad \overline{x}(t)=x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$$

and the Eqs. (14) have the form

$$\frac{d^\alpha x_1(t)}{dt^\alpha}=-2x_1(t)+u(t) \tag{31a}$$

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}x_1(t)+\begin{bmatrix}1\\-1\end{bmatrix}x_2(t)+\begin{bmatrix}0\\-1\end{bmatrix}u(t). \tag{31b}$$

Note that the condition (15) is satisfied since rank $\overline{A}_{22}=$ rank $\begin{bmatrix}1\\-1\end{bmatrix}=n-r=1$. Performing on the array

$$\begin{matrix}\overline{A}_{21}&\overline{A}_{22}&B_2\end{matrix}=\begin{matrix}0&1&0\\1&-1&-1\end{matrix} \tag{32}$$

the elementary row operation $L[2+1\times 1]$, we obtain

$$\begin{array}{ccc} \overline{A}_{31} & \overline{A}_{32} & \overline{B}_2 \\ \widehat{A}_{31} & 0 & \widehat{B}_2 \end{array} = \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & -1 \end{array} \tag{33}$$

and

$$x_2(t)=0, \qquad x_1(t)=u(t). \tag{34}$$

From (31a) and (34) we have

$$\frac{d^\alpha x_1(t)}{dt^\alpha} = -x_1(t)$$

and

$$x_1(t)=\Phi_{10}x_{10}=u(t). \tag{35}$$

Therefore, the consistent initial condition and admissible input should satisfy (35) and $x_{10}=u(0)$, $x_{20}=0$.

Example 2. Check the positivity of the fractional descriptor system (8) with the matrices

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B=\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{36}$$

In this case $q=2$, $n=3$, $m=1$, rank $E=1$

$$\text{rank}\,[Es-A]=\text{rank}\begin{bmatrix} s+2 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}=2 \tag{37}$$

for some $s\in C$ and

$$E_1=[\,1 \quad 0 \quad 0\,], \quad A_1=[\,-2 \quad 0 \quad 0\,], \quad B_1=[1], \qquad A_2=[\,0 \quad 1 \quad -1\,], \quad B_2=[-1]. \tag{38}$$

Note that in this case the condition (15) is not satisfied since $q=2<n=3$. The matrix

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \overline{E}_1=[1], \qquad \overline{x}=x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and Eqs. (23a) and (14b) have the forms

$$\frac{d^\alpha x_1(t)}{dt^\alpha}=-2x_1(t)+u(t), \qquad x_2(t)=x_3(t)+u(t). \tag{39}$$

From (39) it follows that the fractional descriptor system with (36) is positive for any $x_3(t)\geq 0$ and $u(t)\geq 0$, $t\geq 0$ and consistent initial condition $x_{10}\geq 0$ and $x_{20}=x_{30}+u(0)$.

4. Concluding remarks

Positive fractional continuous-time linear systems with a singular pencil have been addressed. A method for checking the positivity and finding the solution to the positive fractional descriptor continuous-time linear systems with a singular pencil has been proposed. The method is based on reduction of the fractional descriptor systems by the use of elementary row and column operations to the equivalent standard systems with some algebraic constraints on state variables and inputs. Necessary and sufficient conditions for the positivity of the fractional descriptor systems have been established. Efficiency of the proposed method has been demonstrated on numerical examples. The considerations can be easily extended to positive fractional discrete-time linear systems with singular pencils.

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