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Pole-free vs. stable-pole designs of minimum variance control for nonsquare LTI MIMO systems

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Abstract. This paper takes advantage of nonuniqueness of the inverse problem for nonsquare transfer function matrices of multivariable systems in order to select such poles, if any, of a minimum variance control system that can either guarantee its closed-loop stability or provide (a sort of) robustness to the control system. As a result, new pole-free and stable-pole MVC designs are offered for nonsquare LTI MIMO systems, the most general of them utilizing the Smith-factorization approach and the so-called control zeros. The new designs contribute to an illustration and extension of the Davison's theory of (nonsquare) minimum phase systems, in that the lack or presence of (appropriate) control zeros can provide a required performance to the MVC system. Simulation examples in the Matlab/Simulink environment confirm the potential of the control zeros and their impact on redefinition of the minimum phase property

Key words: minimum variance control, multivariable control, inverses of polynomial matrices, control zeros, minimum phase systems.

1. Introduction

Inverse-model control (IMC) of nonsquare LTI MIMO systems may present a real research challenge due to an infinite number of solutions to the underlying inverse problem for a nonsquare polynomial matrix [1–3]. In this paper, a sort of IMC, that is minimum variance control (MVC), is analyzed and a series of new results are offered or recalled from conference presentations.

It is well known that the MVC problem, originally formulated and solved for LTI discrete-time systems [4–9], is seldom used in practice mainly due to the lack of robustness and its instability for nonminimum phase systems. Nonetheless, an important inheritance of the original MVC research is at least twofold. Firstly, it has triggered the development of celebrated predictive control strategies [2, 10–17]. Secondly, it has contributed to redefining the *minimum phase* property. In the SISO case, minimum phase systems were defined as those whose transfer function zeros lie strictly inside the unit disk, or those 'stably invertible', or in other words those systems for which MVC is asymptotically stable. This redefinition, probably due to the Åström's group [8, 9], has soon been extended to the square MIMO case [4–6], involving the transmission zeros [18–24]. This has later turned attention to the MVC problem for *nonsquare* LTI MIMO systems [16, 25], including the continuous-time case [2, 15, 26, 27], giving rise to the introduction of *new* multivariable zeros, i.e. the so-called *control zeros*, defined as poles of an inverse transfer function matrix of the system [15, 16, 25, 28]. Control zeros differ from many other multivariable zeros spread throughout the literature [22, 29–35] and aiming at system characterization rather than control.

Control zeros are an intriguing extension of transmission zeros for nonsquare LTI MIMO systems under inverse-model control, in particular MVC. Like the transmission zeros for SISO and square MIMO systems, the control zeros are related to the stabilizing potential of MVC, and, in the input-output modeling framework considered, are generated by (poles of) a generalized inverse of the 'numerator' polynomial matrix $B(.)$ of a system transfer function matrix. Originally, the unique, so-called T-inverse, being the minimum-norm right or least-squares left inverse involving the regular (rather than conjugate) transpose of the polynomial matrix [2, 15, 36], was employed in the specific case of full normal rank systems [2, 15]. The associated control zeros were later called *control zeros type 1* [15, 16, 25], as opposed to an infinite number of *control zeros type 2* [2, 28, 37] generated by a myriad of right/left polynomial matrix inverses.

Selection of 'good' sets of control zeros for control purposes presents a real problem, so we have firstly started from the *pole-free* MVC design case, thus eliminating control zeros and implying the *structural stability* of the closed-loop control system. In our MVC design approach [38–42] based on the extreme points and extreme directions method [38, 40, 42–46], we have offered a pole-free inverse of the polynomial matrix $B(.)$ so that no control zeros appear [39, 42], except when transmission zeros are (nongenerically) present, in which case the extreme points and extreme directions method does not hold. In the contribution of this paper, the Smith factorization of the polynomial matrix $B(.)$ can cope with the problem even when the system has (stable) transmission zeros. The above approaches to the *pole-free* and *stable-pole* MVC designs for LTI MIMO systems [39, 47] constitute an illustration of the transmission zeros-based Davison's definition of the minimum phase property for nonsquare systems [48].

Secondly, we have tackled the problem of a stable-pole MVC design in case an inverse of the polynomial matrix $B(.)$ is not pole-free, that is control zeros do appear in the system

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[39, 47]. To this end, we have introduced minimum-energy MVC, in which selection of stable poles of an inverse of $B(.)$, that is stable control zeros, aims at robustification of MVC [49]. New generalizing results in this respect, in particular concerning the Smith-factorization MVC design, are presented in this paper. These also indicate the role of the control zeros in MVC-related designs, thus contributing to the extension of the Davison's theory of minimum phase systems. Therefore, the theory of control zeros can not only supplement but also compete with the theory of transmission/invariant zeros and the associated theory of minimum phase systems.

The remainder of this paper is organized as follows. The system representations are reviewed in Sec. 2. In Sec. 3, closed-loop discrete-time minimum variance control is recalled and new analytical expressions for various inverses of polynomial matrices are offered, giving rise to the generation of a plethora of control zeros. MVC design methods of Sec. 4 pursue the robustification of this control strategy. A new inverse of a nonsquare polynomial matrix based on the Smith-factorization approach is also offered in that Section. A rationale for the extension of the Davison's definition of minimum phase systems is presented in Sec. 5, culminated with a new, generalized definition of the minimum phase property. Simulation examples of Sec. 6 indicate favorable properties of the new methods in terms of their contribution to a more robust MVC design. New results of the paper are summarized in the conclusions of Sec. 7.

2. System representations

Consider an n_u-input n_y-output linear time-invariant (LTI) discrete-time system with the input $u(t)$ and the output $y(t)$, described by possibly rectangular (nonzero) transfer function matrix $G \in \mathbb{R}^{n_y \times n_u}(z)$ in the complex operator z. The transfer function matrix can be represented in the matrix fraction description (MFD) form $G(z) = A^{-1}(z)B(z)$, where the left coprime polynomial matrices $A \in \mathbb{R}^{n_y \times n_y}[z]$ and $B \in \mathbb{R}^{n_y \times n_u}[z]$ can be given in form $A(z) = z^n I_{n_y} + \ldots + a_n$ and $B(z) = z^m b_0 + \ldots + b_m$, respectively, where n and m are the orders of the respective matrix polynomials and I_{n_y} is the identity n_y-matrix. An alternative MFD form $G(z) = \widetilde{B}(z)\widetilde{A}^{-1}(z)$, involving right coprime $\widetilde{A} \in \mathbb{R}^{n_u \times n_u}[z]$ and $\widetilde{B} \in \mathbb{R}^{n_y \times n_u}[z]$, can also be tractable here but in a less convenient way [16]. Algorithms for calculation of the MFDs are known and software packages in the MATLAB's Polynomial Toolbox® are available. Unless necessary, we will not discriminate between $\underline{A}(z^{-1}) = I_{n_y} + \ldots + \underline{a}_n z^{-n}$ and $A(z) = z^n \underline{A}(z^{-1})$, nor between $\underline{B}(z^{-1}) = \underline{b}_0 + \ldots + \underline{b}_m z^{-m}$ and $B(z) = z^m \underline{B}(z^{-1})$ with $G(z) = A^{-1}(z)B(z) = z^{-d}\underline{A}^{-1}(z^{-1})\underline{B}(z^{-1})$, where $d = n - m$ is the time delay of the system. In the sequel, we will assume for clarity that $B(z)$ is of full normal rank; a more general case of $B(z)$ being of non-full normal rank can also be tractable [16]. Let us finally concentrate on the case when normal rank of $B(z)$ is n_y ('symmetrical' considerations can be made for normal rank n_u). The first MFD form can be directly obtained from the AR(I)X/AR(I)MAX model of a system $\underline{A}(q^{-1})y(t) = q^{-d}\underline{B}(q^{-1})u(t) + [\underline{C}(q^{-1})/\underline{D}(q^{-1})]v(t)$, where q^{-1} is the backward shift operator and $v(t) \in \mathbb{R}^{n_y}$ is the uncorrelated zero-mean disturbance at (discrete) time t; A and B as well as A and $C \in \mathbb{R}^{n_y \times n_y}[z]$ are relatively prime polynomial matrices, with (stable) $\underline{C}(z^{-1}) = \underline{c}_0 + \ldots + \underline{c}_k z^{-k}$ and $k \leq n$, and the $\underline{D}$ polynomial in the z^{-1}-domain is often equal to $1 - z^{-1}$ (or to unity in the discrete-time MVC considerations). In the sequel, we will also use the operator $w = z^{-1}$ (or $w = q^{-1}$), whose correspondence to the s operator for continuous-time systems has pioneeringly been explored in Ref. [26].

The familiar Smith-McMillan form $S_M(w)$ of $G(w) = w^d \underline{A}^{-1}(w)\underline{B}(w)$ (as a special case of the MFD factorization [18]) is given by $G(w) = U(w)S_M(w)V(w)$, where $U \in \mathbb{R}^{n_y \times n_y}[w]$ and $V \in \mathbb{R}^{n_u \times n_u}[w]$ are unimodular and the pencil $S_M \in \mathbb{R}^{n_y \times n_u}(w)$ is of the form

$$S_M(w) = \begin{bmatrix} M_{r \times r} & 0_{r \times (n_u - r)} \\ 0_{(n_y - r) \times r} & 0_{(n_y - r) \times (n_u - r)} \end{bmatrix}, \tag{1}$$

with $M(w) = diag(\varepsilon_1/\psi_1, \varepsilon_2/\psi_2, \ldots, \varepsilon_r/\psi_r)$, where $\varepsilon_i(w)$ and $\psi_i(w)$, $i = 1, \ldots, r$ (with r being the normal rank of $G(w)$), are monic coprime polynomials such that $\varepsilon_i(w)$ divides $\varepsilon_{i+1}(w)$, $i = 1, \ldots, r-1$, and $\psi_i(w)$ divides $\psi_{i-1}(w)$, $i = 2, \ldots, r$. In particular, the Smith form is given by the appropriate pencil $S(w)$, with $M(w) = diag(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$, often associated with Smith zeros or transmission zeros [34]. The polynomials $\varepsilon_i(w)$ are often called the invariant factors of $G(w)$ and their product $\varepsilon(w) = \Pi_1^r \varepsilon_i(w)$ is sometimes referred to as the zero polynomial of $G(w)$.

In the MVC framework, we consider the ARMAX system description

$$\underline{A}(q^{-1})y(t) = q^{-d}\underline{B}(q^{-1})u(t) + \underline{C}(q^{-1})v(t). \tag{2}$$

For the general purposes and for duality with the continuous-time case, we use here the ARMAX model, even though it is well known that the $\underline{C}(q^{-1})$ polynomial matrix of disturbance parameters is usually in control engineering practice unlikely to be effectively estimated (and is often used as a control design, observer polynomial matrix instead).

All the results to follow can be dualized for continuous-time systems described by a Laplace-operator model analogous to Eq. (2). This can be enabled owing to the unified, discrete-time/continuous-time MVC framework introduced in Ref. [26].

3. Closed-loop discrete-time minimum variance control

Consider a right-invertible system described by Eq. (2) and assume that the observer (or disturbance-related) polynomial $\underline{C}(q^{-1}) = \underline{c}_0 + \underline{c}_1 q^{-1} + \ldots + \underline{c}_k q^{-k}$ has all roots inside the unit disk. (Note: Similar results can be obtained for left-invertible systems).

Then the general MVC law, minimizing the performance index

$$\min_{u(t)} E\left\{[y(t+d) - y_{ref}(t+d)]^{\mathrm{T}}\,[y(t+d) - y_{ref}(t+d)]\right\}, \tag{3}$$

where $y_{ref}(t+d)$ and $y(t+d) = \underline{\widetilde{C}}^{-1}(q^{-1})[\underline{\widetilde{F}}(q^{-1})\cdot \underline{B}(q^{-1})u(t)+\underline{\widetilde{H}}(q^{-1})y(t)]+\underline{F}(q^{-1})v(t)$ are the output reference/setpoint and the stochastic output predictor, respectively, is of form [2, 15, 37–42, 49]

$$u(t) = \underline{B}^{\mathrm{R}}(q^{-1})\underline{\widetilde{F}}^{-1}(q^{-1})\cdot \left[\underline{\widetilde{C}}(q^{-1})y_{ref}(t+d)-\underline{\widetilde{H}}(q^{-1})y(t)\right]. \tag{4}$$

The appropriate polynomial $(n_y \times n_y)$-matrices $\underline{\widetilde{F}}(q^{-1}) = I_{n_y} + \underline{\widetilde{f}}_1 q^{-1} + \ldots + \underline{\widetilde{f}}_{d-1} q^{-d+1}$ and $\underline{\widetilde{H}}(q^{-1}) = \underline{\widetilde{h}}_0 + \underline{\widetilde{h}}_1 q^{-1} + \ldots + \underline{\widetilde{h}}_{n-1} q^{-n+1}$ are computed from the polynomial matrix identity (called the Diophantine equation)

$$\underline{\widetilde{C}}(q^{-1}) = \underline{\widetilde{F}}(q^{-1})\underline{A}(q^{-1}) + q^{-d}\underline{\widetilde{H}}(q^{-1}), \tag{5}$$

with

$$\underline{\widetilde{C}}(q^{-1})\underline{F}(q^{-1}) = \underline{\widetilde{F}}(q^{-1})\underline{C}(q^{-1}), \tag{6}$$

where

$$\underline{F}(q^{-1}) = I_{n_y} + \underline{f}_1 q^{-1} + \ldots + \underline{f}_{d-1} q^{-d+1},$$

$$\underline{\widetilde{C}}(q^{-1}) = \underline{\widetilde{c}}_0 + \underline{\widetilde{c}}_1 q^{-1} + \ldots + \underline{\widetilde{c}}_k q^{-k}.$$

Remark 1. The closed-loop stability condition for MVC involves stable poles of $B^{\mathrm{R}}(z)$, that is stable control zeros [2, 15, 16, 25, 37], in terms of their location inside the unit disk.

For right-invertible systems, the symbol $\underline{B}^{\mathrm{R}}(q^{-1})$ denotes, in general, an infinite number of right inverses of the numerator polynomial matrix $\underline{B}(q^{-1})$. In an attempt to seek for unique right inverses, possibly minimum-norm ones, we have analyzed [2, 15, 50] various forms of the MVC Eq. (4), which can be treated as a solver of the linear polynomial matrix equation

$$\underline{B}(q^{-1})u(t) = \underline{y}(t), \tag{7}$$

where

$$\underline{y}(t) = \underline{\widetilde{F}}^{-1}(q^{-1})\left[\underline{\widetilde{C}}(q^{-1})y_{ref}(t+d) - \underline{\widetilde{H}}(q^{-1})y(t)\right].$$

The first obvious inverse of $\underline{B}(q^{-1})$ is what we call the T-inverse, that is the minimum-norm right inverse

$$\underline{B}_0^{\mathrm{R}}(.) = \underline{B}(.)^{\mathrm{T}}[\underline{B}(.)\underline{B}(.)^{\mathrm{T}}]^{-1}, \tag{8}$$

whose poles are defined as *control zeros type 1* [2, 15, 27, 28, 37, 50]. However, we have shown in Refs. [38, 40–43, 51] that there are some other right inverses possible, all providing $\underline{B}(.)\underline{B}^{\mathrm{R}}(.) = I_{n_y}$, that is the satisfaction of the MVC criterion (3). The inverses are associated with a general class of solvers for $u(t)$ in the MVC problem [2, 15, 50]

$$[\beta(q^{-1})]\{I_{n_u} + [\beta(q^{-1})]^{\mathrm{R}}[\underline{B}(q^{-1}) - \beta(q^{-1})]\}u(t) = \underline{y}(t), \tag{9}$$

with $\beta(q^{-1})$ being the set of all (full normal rank) matrix subpolynomials of order $0, 1, \ldots, m-1$ of $\underline{B}(q^{-1})$. Note that for $\beta(q^{-1}) = \underline{B}(q^{-1})$ we arrive at Eq. (7).

Applying the minimum-norm right T-inverse again (subindexed by 0) we have defined another, more general class of the right inverses of $\underline{B}(q^{-1})$, which are called the τ-inverses [2, 37, 50]

$$\underline{B}^{\mathrm{R}}(q^{-1}) = \{I_{n_u} + [\beta(q^{-1})]_0^{\mathrm{R}}[\underline{B}(q^{-1}) - \beta(q^{-1})]\}^{-1}[\beta(q^{-1})]_0^{\mathrm{R}} \tag{10}$$

whose poles generate one class of *control zeros type 2* [2, 28, 37, 39, 50, 51]. Note that for $\beta(.) = \underline{B}(.)$ we arrive at the T-inverse of $\underline{B}(.)$.

The numerous inverses and subinverses of $\beta(q^{-1})$ are calculated 'downward' according to formulae (8) and (10), down to the component monomials finally inverted.

Theorem 1. Consider a nonsquare full-normal rank polynomial matrix $\underline{B}(q^{-1}) = \underline{b}_0 + \underline{b}_1 q^{-1} + \ldots + \underline{b}_m q^{-m}$. The total number N_m of the τ-inverses of $\underline{B}(q^{-1})$ is calculated iteratively from the equation

$$N_i = 1 + (i+1)!\sum_{j=1}^{i} \frac{1}{j!(i-j+1)!} N_{j-1}, \qquad i = 1, \ldots, m; \qquad N_0 = 1. \tag{11}$$

Proof. Immediate (by inspection or via fundamental induction and combinatorics arguments).

In the motivating example of Refs. [2, 50], that is for $m = 2$, we have specified all the 12 τ-inverses, giving rise to 12 sets of control zeros type 2.

Let us proceed now to the most intriguing issue related to the family of inverses as in Eq. (10). It is surprising that $\underline{B}(q^{-1})\underline{B}^{\mathrm{R}}(q^{-1}) = I_{n_y}$, with $\underline{B}^{\mathrm{R}}(q^{-1})$ as in Eq. (10), even for *arbitrary* $\beta(q^{-1})$, that is not related to $\underline{B}(q^{-1})$ at all (but, of course, with the adequate matrix dimensions). This way we arrive at the so-called σ-inverses, a number of which is infinite (in spite of the unique minimum-norm right inverse involved). Even though the most general σ-inverses contain the τ-inverses, which in turn include the T-inverse, we will discriminate between the three types of inverses of nonsquare polynomial matrices. The poles of the σ-inverses of $\underline{B}(.)$ generate the second class of control zeros type 2 [2, 39, 47]. Of course, due to an infinite number of σ-inverses, there is an infinite number of sets of control zeros type 2.

Remark 2. Although transmission zeros, if any, make a subset of control zeros (see Subsec. 4.3), we still discriminate between the two notions for 'traditional' reasons.

It is worth emphasizing that the formula (10) with arbitrary $\beta(q^{-1})$ is a general, analytical expression for calculation of right inverses for nonsquare polynomial matrices. Quite similar formula can be given for left inverses. It is interesting to note how stimulating was the MVC framework for the derivation of the new inverses.

4. Pole-free designs of stable MVC

The question arises now whether it is possible to design MVC in a pole-free way, that is to select $\underline{B}^{\mathrm{R}}(q^{-1})$ having no poles, implying that no control zeros appear at all. In this case stable MVC could be obtained without any reference to control zeros but possibly to transmission zeros, if any. Specifically, a pole-free design would *guarantee* the stability of MVC for nonsquare systems without transmission zeros, which is a generic case. Here we present two new solutions for pole-free MVC designs. In contrast, we also offer a stable-pole MVC design, with stable sets of control zeros selected to provide a sort of robustness to MVC, the approach outperforming the pole-free design, at least for a specific system under study.

Remark 3. Note that our pole-free design of $\underline{B}^{\mathrm{R}}(w)$ means that $\underline{B}^{\mathrm{R}}(w)$ is just a matrix polynomial. Also note that, strictly speaking, the term 'pole-free' is considered with respect to the variable $w = z^{-1}$. When translated to the z operator, this will end-up with pole(s) at zero, which, however, does not affect the guaranteed stability of MVC.

Remark 4. The term 'pole-free' (or 'control zero-free') relates to the lack of control zeros other than transmission zeros (see also Remark 2).

4.1. Extreme points and extreme directions (EPED) method. The method is recalled here for solution of the linear matrix polynomial equation [42, 43, 46, 52]

$$K(w)X(w) = P(w), \tag{12}$$

where $K(w) = K_0 + K_1 w + \ldots + K_{n_K} w^{n_K}$ and $P(w) = P_0 + P_1 w + \ldots + P_{n_P} w^{n_P}$ are given $m \times n$ and $m \times p$ polynomial matrices in complex operator w, respectively, and $X(w) = X_0 + X_1 w + \ldots + X_{n_X} w^{n_X}$ is an $n \times p$ polynomial matrix to be found. By equating the powers of w in the formula (12), we obtain an equivalent linear system of equations

$$\overline{K}\,\overline{X} = \overline{P}, \tag{13}$$

where the real matrix

$$\overline{K} = \begin{bmatrix} K_0 & & & 0 \\ K_1 & K_0 & & \\ \vdots & K_1 & \ddots & \\ K_{n_K} & \vdots & & K_0 \\ & K_{n_K} & & K_1 \\ & & \ddots & \vdots \\ 0 & & & K_{n_K} \end{bmatrix}, \tag{14}$$

is referred to as the Sylvester matrix of $K(w)$ of order n_K, with $\widetilde{m} = (n_K + n_X + 1)m$ rows and $\widetilde{n} = (n_X + 1)n$ columns and

$$\overline{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n_X} \end{bmatrix} \in \mathbb{R}^{\widetilde{n} \times p},$$

$$\overline{P} = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n_P} \end{bmatrix} \in \mathbb{R}^{\widetilde{m} \times p}. \tag{15}$$

The problem of finding the matrix polynomial solution $X(w)$ to Eq. (12) has been reduced to finding the real matrix $\overline{X}$ of Eq. (13) for given real matrices $\overline{K}$ and $\overline{P}$ as in Eqs. (14) and (15). The matrix polynomial equation (12) has the solution for $X(w)$ iff $rank \begin{bmatrix} \overline{K} & \overline{P} \end{bmatrix} = rank\, \overline{K}$. After using the Kronecker product the Eq. (13) can be rewritten into the form

$$Ax = b, \tag{16}$$

where

$$A = \overline{K} \otimes I_p \in \mathbb{R}^{\overline{m} \times \overline{n}},$$

$$x = \begin{bmatrix} \overline{x}_1, & \overline{x}_2, & \ldots, & \overline{x}_{\widetilde{n}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\overline{n}},$$

$$b = \begin{bmatrix} \overline{p}_1, & \overline{p}_2, & \ldots, & \overline{p}_{\widetilde{m}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\overline{m}},$$

with $\overline{m} = \widetilde{m}p$, $\overline{n} = \widetilde{n}p$, and $\overline{x}_i$ and $\overline{p}_j$ denote the i-th and j-th rows of $\overline{X}$ and $\overline{P}$, respectively.

Now, the problem of calculation of the set of solutions to Eq. (12) can be reduced to finding the set x satisfying the Eq. (16). Note that if $\widetilde{n} \geq \widetilde{m}$ and $rank\, \overline{K} = \widetilde{m}$, then the matrix A also has full row rank.

Let $S = \{x : Ax = b\}$ be a non-empty set. A point x is an *extreme point* of S iff A can be decomposed into $\begin{bmatrix} B & N \end{bmatrix}$ such that $\det(B) \neq 0$ and $x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$. If $rank\, A = \overline{m}$, then S has at least one extreme point. The number of extreme points is less than or equal to $\dfrac{\overline{n}!}{\overline{m}!(\overline{n} - \overline{m})!}$.

A vector d is an *extreme direction* of S iff A can be decomposed into $\begin{bmatrix} B & N \end{bmatrix}$ such that $\det(B) \neq 0$ and $d = \begin{bmatrix} -B^{-1}a_j \\ e_j \end{bmatrix}$, where a_j is the i-th column of N and e_j is an $\overline{n} - \overline{m}$ vector of zeros except for unity in position i. The set S has at least one extreme direction iff it is unbounded. The maximum number of extreme directions is bounded by $\frac{\overline{n}!}{\overline{m}!(\overline{n} - \overline{m} - 1)!}$.

Let x_1, x_2, …, x_k be the extreme points of S and d_1, d_2, …, d_l be the extreme directions of S. Then every $x \in S$ can be written as $x = \sum_{j=1}^{k} \lambda_j x_j + \sum_{i=1}^{l} \mu_i d_i$, $\sum_{j=1}^{k} \lambda_j = 1$.

Let us now embed the EPED method in the MVC-related framework.

Theorem 2. Consider the MVC control law (4) and let $\underline{B}^{\mathrm{R}}(w) = X(w)$, with $w = z^{-1}$, be a solution of the linear matrix polynomial equation $\underline{B}(w)X(w) = I_{n_y}$. Then the necessary and sufficient condition for the existence of the solution by the EPED method is that the underlying system has no transmission zeros.

Proof. It is well known that the necessary and sufficient condition for the existence of a solution of Eq. (12) is that $nrank \begin{bmatrix} K(w) & P(w) \end{bmatrix} = nrank\, K(w)$, where $nrank$ stands for normal rank. When translated to the MVC framework, the condition $nrank \begin{bmatrix} \underline{B}(w) & I_{n_y} \end{bmatrix} = nrank\, \underline{B}(w)$ implies that the system under MVC has no transmission zeros.

Well, the mathematically elegant EPED method provides a pole-free solution to the MVC problem but it will be shown to be computationally involving and yet its use is limited to systems with no transmission zeros. Therefore, in the next section we offer a new, much simpler method, which is valid for systems with transmission zeros as well. Even if the method is more effective, it is just the EPED method who has turned our attention to pole-free MVC design.

4.2. Smith-factorization approach. In an attempt to essentially reduce the computational burden of the EPED method we introduce yet another effective (and much simpler) approach to stable MVC design.

Consider an n_u-input n_y-output LTI system described by the ARMAX model (2). Put $w = z^{-1}$ and factorize $\underline{B}(w)$ to the Smith form $\underline{B}(w) = U(w)S(w)V(w)$, where $U(w)$ and $V(w)$ are unimodular. Now, $\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)S^{\mathrm{R}}(w)U^{-1}(w)$, with determinants of $U(w)$ and $V(w)$ being independent of w, that is possible instability of an inverse polynomial matrix $\underline{B}^{\mathrm{R}}(w)$ being related to $S^{\mathrm{R}}(w)$ only.

Theorem 3. Consider the problem of MVC of an LTI n_u-input n_y-output system described by the ARMAX model (2), with $\underline{B}(z^{-1})$ being of full normal rank n_y. Use the Smith factorization and obtain the matrix $\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)S^{\mathrm{R}}(w)U^{-1}(w)$, with $w = z^{-1}$ and $U(w)$ and $V(w)$ being unimodular. Then applying the minimum-norm right T-inverse $S_0^{\mathrm{R}}(w) = S^{\mathrm{T}}(w)\left[S(w)S^{\mathrm{T}}(w)\right]^{-1}$ guarantees stable pole-free MVC design for systems without transmission zeros and stable MVC design for systems with stable transmission zeros.

Proof. Performing the Smith factorization for $\underline{B}(w)$ one obtains $\underline{B}(w) = U(w)S(w)V(w)$, where $U(w)$ and $V(w)$ are unimodular. Now, $\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)S^{\mathrm{R}}(w)U^{-1}(w)$, with determinants of $U(w)$ and $V(w)$ being independent of w, that is possible instability of an inverse polynomial matrix $\underline{B}^{\mathrm{R}}(w)$ being related to $S^{\mathrm{R}}(w)$ only. Since in general $S(w) = \begin{bmatrix} diag(\varepsilon_1, \ldots, \varepsilon_{n_y}) & 0_{n_y \times (n_u - n_y)} \end{bmatrix} = S_{tz}(w)\underline{S}$, where $S_{tz}(w) = diag(\varepsilon_1, \ldots, \varepsilon_{n_y})$ includes transmission zeros and $\underline{S} = \begin{bmatrix} I_{n_y} & 0_{n_y \times (n_u - n_y)} \end{bmatrix}$, we have $\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)\underline{S}^{\mathrm{R}}S_{tz}^{-1}(w)U^{-1}(w)$. Now $\underline{S}_0^{\mathrm{R}} = \underline{S}^{\mathrm{T}}\left[\underline{S}\,\underline{S}^{\mathrm{T}}\right]^{-1} = \begin{bmatrix} I_{n_y} & 0_{n_y \times (n_u - n_y)} \end{bmatrix}^{\mathrm{T}}$ and the result follows immediately.

4.3. Stable Smith-factorization MVC design with arbitrary degrees of freedom. In the previous section, stable MVC designs have been obtained without any reference to possible infinite number of degrees of freedom, which can be of interest in robustness considerations for MVC. Even though the unimodular matrices involved are nonunique, possible use of the resulted degrees of freedom is rather difficult due to the constraints imposed on matrix determinants. Here we present a simple Smith-factorization approach to stable MVC design with arbitrary degrees of freedom. Recall $\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)\underline{S}^{\mathrm{R}}S_{tz}^{-1}(w)U^{-1}(w)$. With a specific form of $\underline{S} = \begin{bmatrix} I_{n_y} & 0_{n_y \times (n_u - n_y)} \end{bmatrix}$, we can immediately offer its arbitrary right inverse $\underline{S}^{\mathrm{R}} = \underline{S}^{\mathrm{R}}(w) = \begin{bmatrix} I_{n_y} \\ L(w) \end{bmatrix}$, where $L(w)$ is any polynomial matrix of the appropriate dimensions. A general form of that matrix can be $L(w) = \{l_{ij}(w)\}$, $i = 1, ..., n_u - n_y$, $j = 1, ..., n_y$, with $l_{ij}(w) = l_{ij}^{(0)} + l_{ij}^{(1)}w + \ldots + l_{ij}^{(m_{ij})}w^{m_{ij}}$ and m_{ij} can be an arbitrary natural number selected by the designer.

Remark 5. Note that the solution $\underline{S}^{\mathrm{R}} = \underline{S}^{\mathrm{R}}(w) = \begin{bmatrix} I_{n_y} \\ L(w) \end{bmatrix}$ can as well be obtained by the EPED method, with $K(w) = \underline{S}$, $X(w) = \underline{S}^{\mathrm{R}}$ and $P(w) = I_{n_y}$.

Remark 6. It is worth mentioning once more that all the above MVC designs guarantee closed-loop stability of MVC both in case of the lack of transmission zeros and under stable transmission zeros, with possible control zeros (other than transmission zeros) totally eliminated.

Remark 7. It is the right T-inverse applied to the matrix $\underline{S}$ that enables to eliminate control zeros. Applying some other inverses, that is τ- and σ-inverses, usually ends up with control zeros. However, our simulating experience shows that it is sometimes possible to choose such a matrix polynomial $\beta(z^{-1})$ that pole-free σ-inverse(s) of $\underline{B}(z^{-1})$ can be obtained. A general selection of such σ-inverses is a very difficult and still open problem. Therefore it seems that the new inverse of the $\underline{B}(z^{-1})$ matrix, based on the Smith factorization can be more useful in stable MVC design.

Remark 8. The question arises whether in some cases pole-free, that is control zero-free MVC designs could be inferior to the synthesis allowing for (stable) poles of the closed-loop MVC system, that is (stable) control zeros selected to provide e.g. a sort of robustness to MVC. Such a solution is presented in Subsec. 4.5.

Remark 9. It is in general possible in the above stable Smith-factorization MVC design to select $L(w)$ as a (stable) rational matrix or, in a technically simpler way, as a matrix with all its elements being (stable) rational transfer functions, with an infinite number of degrees of freedom possible to obtain from degree(s) of the transfer functions. Then the poles of those transfer functions are the poles of a closed-loop MVC system, that is the control zeros (well, in addition to possible transmission zeros treated above). Alternatively, the control zeros (together with transmission zeros) can be generated as poles of σ-inverses of $\underline{B}(z^{-1})$, with an infinite number of degrees of freedom obtained from arbitrary matrix polynomials $\beta(z^{-1})$.

4.4. New inverse of a nonsquare polynomial matrix. Concluding the Smith-factorization approach to design of (stable) MVC, we can offer yet another, new, general right inverse of a nonsquare polynomial matrix $\underline{B}(z^{-1})$, which can be competitive to τ- and σ-inverses and which we call an S-inverse. The new result, following immediately from Theorem 3 and Subsec. 4.3, is given in form of

Corollary 1. Consider a polynomial $n_y \times n_u$ matrix $\underline{B}(w)$ of full normal rank n_y, with $w = z^{-1}$, under the Smith factorization $\underline{B}(w) = U(w)S(w)V(w)$, where $U(w)$ and $V(w)$ are unimodular and $S(w) = \begin{bmatrix} diag(\varepsilon_1, \ldots, \varepsilon_{n_y}) & 0_{n_y \times (n_u - n_y)} \end{bmatrix}$. Then a general right inverse of $\underline{B}(w)$ can be given as

$$\underline{B}^{\mathrm{R}}(w) = V^{-1}(w)\underline{S}^{\mathrm{R}}(w)S_{tz}^{-1}(w)U^{-1}(w), \tag{17}$$

where $S_{tz}(w) = diag(\varepsilon_1, \ldots, \varepsilon_{n_y})$, with ε_i being the invariant factors, and $\underline{S}^{\mathrm{R}}(w) = \begin{bmatrix} I_{n_y} \\ L(w) \end{bmatrix}$, where $L(w)$ is an arbitrary rational matrix of the appropriate dimensions (providing the degrees of freedom).

Remark 10. Note that the control zeros are the poles of $S^{\mathrm{R}}(w) = \underline{S}^{\mathrm{R}}(w)S_{tz}^{-1}(w)$, which confirms that the transmission zeros make a subset of control zeros. However, for control zeros generated by τ- and σ-inverses, this can be confirmed in simulations only, so far.

Remark 11. So far, no relation between σ- and S-inverses could be found. Apparently, these are two distinct classes of inverses of nonsquare polynomial matrices.

It would be interesting now to compare performances of a stable-pole MVC design, admitting (stable) control zeros, with the previous pole-free designs, eliminating control zeros.

4.5. Minimum-energy design. MVC is seldom used in practice, mainly due to the lack of robustness. In particular, excursions of control variable(s) beyond admissible technological limits can make the MVC strategy inapplicable in control engineering tasks. One possible way to robustify MVC is to impose an energy constraint on the control signal. A reasonable approach is to minimize the criterion (3) subject to the minimum energy constraint for the control signal (compare [53–56])

$$\min_{\{\zeta\}} \sum_{t=0}^{\infty} u^{\mathrm{T}}(t)u(t), \tag{18}$$

over all possible sets $\{\zeta\}$ of control zeros. The 'only' problem is an infinite number of sets of control zeros. However, instead of minimizing the energy with respect to all sets of control zeros we can run the minimization over sets of polynomial matrices $\beta(q^{-1})$, or rather their parameter matrices (for a specified order m). Although computationally very involving, in general, the energy minimization problem can now be solved in a conceptually easy way in the Matlab/Simulink environment. It is interesting to observe that the minimum energy criterion (18) rules out unstable MVC, so that unstable control zeros are out of interest. Thus, computational burden of the minimization procedure can be reduced by skipping over such sets of the parameter matrices that generate unstable control zero(s), the concept being a simple realization of a constrained optimization procedure.

Note that instead of employing σ-inverses and running the minimization procedure over sets of polynomial matrices $\beta(q^{-1})$ we can, alternatively, apply S-inverses and perform the minimization over sets of rational matrices $L(q^{-1})$.

The minimum-energy MVC design provides a tool for robustification of MVC on the one hand, and yet another justification for the introduction of control zeros on the other. In fact, the minimal energy will be shown in simulations to be obtained for a certain set of (stable) control zeros rather than for the control zero-free case. Formal confirmation of this simulation observation is a challenging future research task.

5. Minimum phase LTI MIMO systems

It is well known that nonsquare LTI MIMO systems generically have no transmission zeros, which implies, according to the Davison's theory [48], that they are, generically, minimum phase. Unstable transmission zero(s), if any, can make the system nonminimum phase. The theory, extended to invariant zeros, has been considered unquestionable. However, the theory has by no means been related to control of a system. It is well known that, for square systems (including SISO ones), the minimum phase behavior related to stable transmission zeros has later been translated into stable MVC or stable inverse-model control. In an input-output system description, this has associated the minimum phase property with stable inverse of a numerator polynomial (matrix) $\underline{B}(.)$, that is stable transmission zeros.

We have proposed to apply right the same paradigm to nonsquare systems, with MVC distinguished for the purpose again as the maximum-accuracy and maximum-speed control [28]. However, a plethora of generalized inverses of a (nonsquare) polynomial matrix $\underline{B}(q^{-1})$ has raised the question of usefulness of the associated sets of zeros, which we have called control zeros [2, 15, 27, 28, 37, 39, 50]. Firstly, we have introduced here pole-free MVC designs, in which case we can totally eliminate control zeros, except of transmission zeros, if any, so that the transmission zeros decide on the minimum/nonminimum phase behavior. This is just the case covered by the Davison's theory. Secondly, we will attempt at answering the question: "Having the tools for getting rid of control zeros, why not completing the game, joining the Davison's theory and forgetting about the control zeros?". To this end, we will try to find if there exist such sets of control zeros for which the performance of MVC might be 'better' than that for the case of the absence of control zeros. There is one possible factor that can decide upon 'betterness' of one MVC solution over the other and this is robustness. We will thus compare various MVC solutions, in terms of various polynomial matrix inverses and various sets of control zeros, in respect of energy of the control action. In particular, we will examine the minimum-energy MVC design (as excessive excursions of control variable(s) are the well-known disadvantage of the MVC strategy). In our simulation example of Sec. 6, the minimum-energy MVC solution will be obtained for a specific set of (stable) control zeros and the minimal energy will be found lower than that for the pole-free MVC design. Thus, a proper selection of control zeros can contribute to (a sort of) robustness of MVC, the new result finally leading to counterexampling the Davison's theory. In fact, control zeros can be really useful and a definition of the minimum phase property should account for control zeros, being a generalization of transmission zeros. In other words, the minimum phase property should be defined with respect to control zeros. Here we present a general definition of a minimum phase system governed by its (nonzero) transfer function matrix.

Definition 1. An LTI MIMO system is called *minimum phase* iff it is stably invertible in terms of any generalized inverse of its transfer function matrix; otherwise the system is called *nonminimum phase*.

Remark 12. The above definition suggests that, from a number of possible inverses of the system transfer function matrix, we can seek for stable inverse(s), which is illustrated for a rather impractical case of a non-full normal rank system in Example 1.

In order to be more practical (compare [16, 18]) we introduce a particular variant of the above definition for full normal rank systems.

Definition 2. A full normal rank LTI MIMO system is called *minimum phase with respect to a specific set of its control zeros* (including transmission zeros) iff this set of control zeros is strictly stable (that is it lies strictly within the unit disk for discrete-time systems or strictly in the left half-plane for continuous-time ones). Otherwise the system is called *nonminimum phase with respect to the specific set of control zeros*.

Remark 13. Definitions 1 and 2 are valid for both square and nonsquare systems, of course.

Illustrations of Definition 2 can be found in Examples 2 and 3.

6. Simulation examples

Example 1.
Consider a square LTI MIMO system with the non-full normal rank matrix

$$\underline{B}(w) = \begin{bmatrix} 0.38w^2+2.41w+2.35 & 1.00w^2+2.90w+2.00 & 0.17w^2+1.35w+1.50 \\ 0.47w^2+1.91w+0.74 & 1.15w^2+2.05w+0.70 & 0.19w^2+1.19w+0.50 \\ 1.20w^2+5.80w+2.70 & 3.00w^2+6.50w+3.00 & 0.50w^2+3.40w+2.00 \end{bmatrix}.$$

The Smith factorization of $\underline{B}(w)$ produces the matrix

$$\underline{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which shows that the system has no transmission zeros (and no control zeros). The Moore-Penrose inverse of $\underline{S}$ is just $\underline{S}^{\#} = \underline{S}$, so the system is minimum phase according to Definition 1. But on the other hand, some other inverses of $\underline{S}$ can be obtained as

$$\underline{S}^{inv_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & L(w) & \end{bmatrix}$$

or

$$\underline{S}^{inv_2} = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & L(w) \\ 0 & 0 & \end{bmatrix},$$

both satisfying only two Moore-Penrose conditions, where $L(w)$ is an arbitrary rational matrix of the appropriate dimensions, whose selection can lead to generation of arbitrary control zeros determining the minimum/nonminimum phase behavior of the system.

Quite similar results can be obtained for a nonsquare LTI MIMO system with non-full normal rank matrix $B(z)$.

Example 2.
Extreme points and extreme directions method.
Consider an unstable three-input two-output second order system described by the ARMAX model (2), with

$$\underline{B}(w) = \begin{bmatrix} 0.1 & 0.9w^2-0.3w+1.6 & -0.7w \\ -1.4w^2+0.6w-1.3 & -1.3 & -0.6w^2 \end{bmatrix},$$

$$\underline{A}(w) = \begin{bmatrix} 0.1w^2-0.4w+3.0 & 0.2w^2-0.3w+2.0 \\ 0.4w^2+0.5w+1.0 & 0.1w^2+0.2w+0.8 \end{bmatrix},$$

$$\underline{C}(w) = \begin{bmatrix} -0.2w^2-0.2w+3.0 & 0.1w^2-0.3w+2.0 \\ -0.1w^2+0.4w+1.0 & 0.1w^2+0.1w+0.8 \end{bmatrix}$$

and $d = 2$, with $w = q^{-1}$. Inspection of an unstable set of control zeros type 1, obtained on a basis of the T-inverse, may suggest that the system is nonminimum phase (with respect to that set of zeros), which is confirmed by the examination of unstable MV/perfect control of the system. The unstable sets of control zeros type 2, obtained via twelve τ-inverses, also seem to confirm the nonminimum phase behavior of the analyzed system (with respect to those sets of control zeros).

However, stable MV/perfect control solutions as in Eq. (4), can be immediately obtained from the pole-free design presented in Subsec. 4.1.

Specifically, for $n_x = 4$ the solutions involve e.g. the Sylvester matrix $\overline{K}$ of form presented in the Appendix A.

For our example the number of extreme points is less than or equal to 435 and the maximum number of extreme directions is bounded by 870.

From a plethora of solutions generated by the extreme points and extreme directions method we have chosen the following one $x = 2.3x_1|_{N_1} - 1.3x_2|_{N_2} + 0.28d_1|_{N_3,e_{31}} - 0.05d_2|_{N_3,e_{32}} + 0.11d_3|_{N_4,e_{41}} - 0.09d_4|_{N_4,e_{42}}$, where $N_1 = \begin{bmatrix} a_5 & a_{26} \end{bmatrix}$, $N_2 = \begin{bmatrix} a_7 & a_{30} \end{bmatrix}$, $N_3 = \begin{bmatrix} a_{13} & a_{16} \end{bmatrix}$, $N_4 = \begin{bmatrix} a_9 & a_{20} \end{bmatrix}$, $e_{31} = e_{41} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $e_{32} = e_{42} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Finally, the specific solution $X(w) = \underline{B}^{\mathrm{R}}(w)$ is of form given in the Appendix B.

Now, for $y_{1ref} = 0.5$ and $y_{2ref} = -0.4$ and for the zero-mean uncorrelated disturbance vector $v(t)$ with $var\{v_1(t)\} = 2.46e{-}4$ and $var\{v_2(t)\} = 2.25e{-}4$, the MV/perfect control outputs $y_1(t)$ and $y_2(t)$ are shown in Figs. 1 and 2, respectively, and the selected inputs $u_1(t)$ and $u_2(t)$ are plotted in Figs. 3 and 4, respectively.

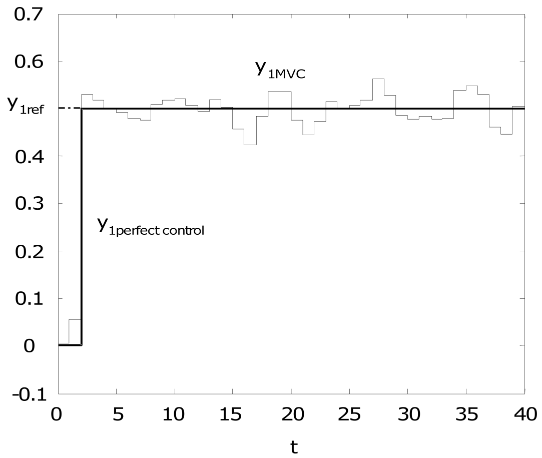

Fig. 1. MVC and perfect control: plots of the output 1 vs. output reference for Example 2

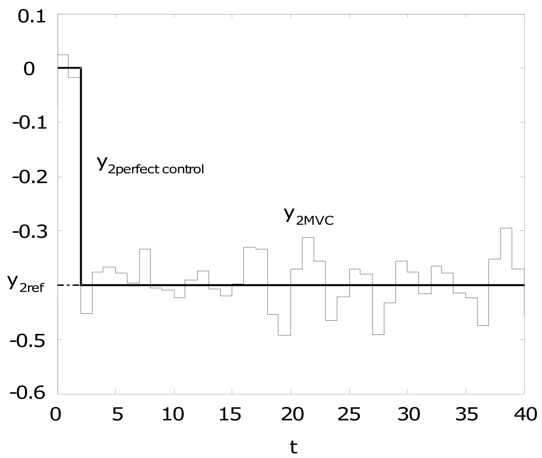

Fig. 2. MVC and perfect control: plots of the output 2 vs. output reference for Example 2

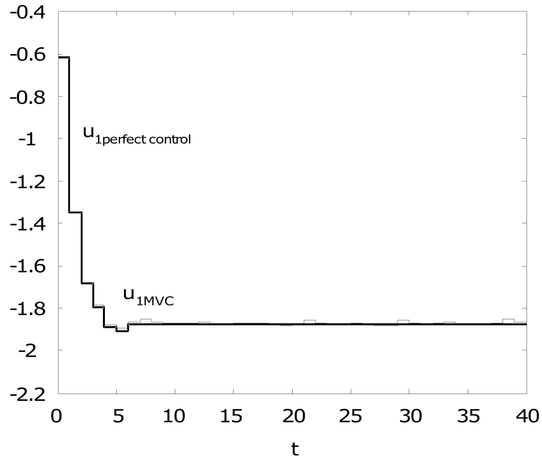

Fig. 3. MVC and perfect control: plots of the input 1 for Example 2

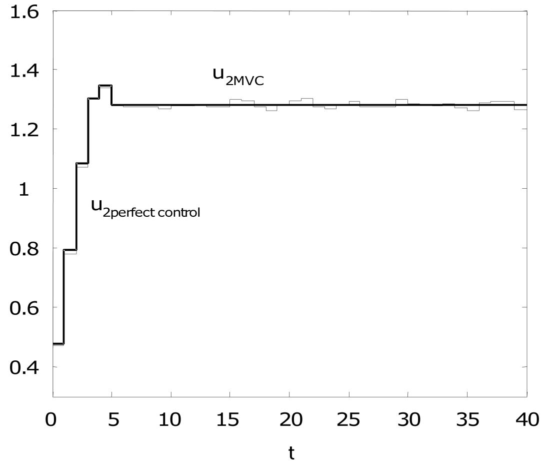

Fig. 4. MVC and perfect control: plots of the input 2 for Example 2

Example 3.
Smith-factorization and minimum-energy approaches.
Consider a three-input two-output unstable system described by the ARMAX model (2) with

$$\underline{B}(w) = \begin{bmatrix} -0.10w^2 + 0.90w + 1.00 & \dots \\ -0.05w^2 + 0.40w + 1.00 & \dots \end{bmatrix}$$

$$\begin{matrix} 0.40w^2 + 1.00w & -0.02w^2 + 0.10w + 1.00 \\ 1.00w^2 & -0.10w^2 + 0.80w + 2.00 \end{matrix}\Bigg] ,$$

$$\underline{A}(w) = \begin{bmatrix} 1.00w^2 + 1.00w + 1.00 & 1.00w^2 + 0.50w + 1.00 \\ 0.10w^2 + 0.50w + 1.00 & 0.20w^2 + 0.30w + 1.00 \end{bmatrix},$$

$$\underline{C}(w) = \begin{bmatrix} 0.10w^2 + 0.20w + 1.00 & 0.20w^2 - 0.30w + 1.00 \\ 0.10w^2 - 0.40w + 1.00 & 0.10w^2 - 0.60w + 1.00 \end{bmatrix},$$

$d = 2$, $var\{v_1(t)\} = 9.88e - 6$ and $var\{v_2(t)\} = 9.88e - 6$. The system has one stable transmission zero at $z = 0.1$ but it can be considered nonminimum phase with respect to all unstable 13 sets of control zeros produced by 13 sets of the T- and τ-inverses. Still, by the selection of a proper polynomial matrix $\beta(q^{-1})$ we can arrive at such σ-inverse(s) that generate stable control zeros, with respect to which the system is minimum phase. This can be obtained through running the energy minimization procedure in the Matlab/Simulink package, returning as follows:

$$\beta(w) = \begin{bmatrix} -0.0662w - 20.6614 & \dots \\ -0.1029w + 1.5865 & \dots \end{bmatrix}$$

$$\begin{matrix} 2.6090w + 1.8460 & 57.2021w + 81.0039 \\ 1.9270w - 0.1948 & -0.0714w + 3.0760 \end{matrix} \Bigg],$$

control_zeros: stable, transmission_zero: stable,
min_energy= 1319.4187.

Stable MVC performance for $y_{1ref} = 1.5$ and $y_{2ref} = 2$ can be observed in Figs. 5 and 6, with the control inputs and outputs plotted respectively.

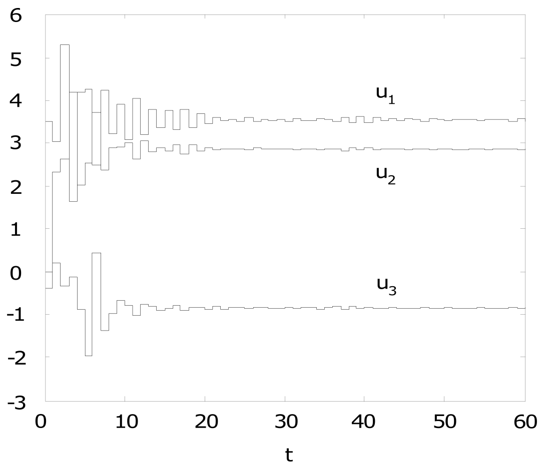

Fig. 5. Minimum-energy MVC inputs for Example 3

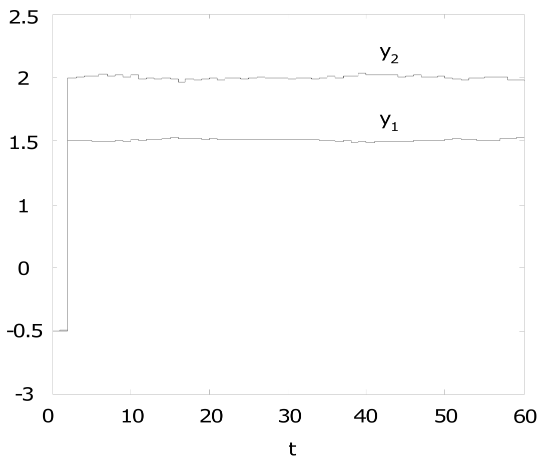

Fig. 6. Minimum-energy MVC outputs for Example 3

It is interesting to compare this performance with that for the pole-free MVC design based on the Smith factorization, where control zeros are eliminated. An example of what is returned by the procedure is as follows:

control_zeros: none, transmission_zero: stable,
energy=16074.4435.

It can be seen that the energy of the control input is now remarkably higher than for the previous, optimal case involving the selected stable control zeros. The performance difference between the two cases is quite high and the bequeath is clear: some set(s) of stable control zeros can provide more robust MVC than that for the pole-free case. Thus, it is sometimes an interest not to get rid of control zeros.

It is worth mentioning that the MVC energy minimization procedure was developed in the Matlab/Simulink environment, with an application of the *fminsearch* function and verification by means of the *GAOT* package.

7. Conclusions

Nonuniqueness of a solution to the inverse problem for nonsquare transfer function matrices of multivariable systems provides an infinite number of degrees of freedom, which can be useful in design of inverse-model control, in particular MVC, for nonsquare LTI MIMO systems. Specifically, selection of poles, if any, of the closed-loop MVC, that is poles of an inverse transfer function matrix of the nonsquare system, can yield a variety of control performances. In this paper, new pole-free and stable-pole designs are offered for robust MVC of nonsquare LTI MIMO systems. The introduced Smith-factorization approach is shown to outperform the earlier contributions. The relationship of the designs with multivariable zeros and the minimum phase property is emphasized.

It has been shown that when dealing with the minimum/nonminimum phase property for nonsquare LTI MIMO systems it is necessary to consider a control environment, in terms of inverse model control, in particular MVC. In fact, the stability/instability of MVC for nonsquare systems can be related, like for square MIMO and SISO systems, with the minimum/nonminimum phase behavior. Therefore, the control zeros, a new generalization of the transmission zeros, are necessary to be considered to this end. It has been shown how control zeros can possibly be eliminated by means of the pole-free MVC designs, in which particular case we end up with the Davison's theory of transmission zeros and minimum phase systems. But on the other hand, usefulness of selected sets of stable control zeros has been shown for the stable-pole minimum-energy MVC design. Consequently, a new definition of the minimum phase property has been offered. The control zeros-oriented definition presents an extension of the Davison's theory of minimum phase systems. Thus, it is sometimes an interest not to eliminate control zeros. Well, at least the awareness of possible occurrence of control zeros in MVC-related tasks is apparently welcome. The series of the new results provide a new light to understanding of the problem of control zeros of nonsquare LTI MIMO systems and their application in stable/robust MVC designs.

Appendix A

$$\overline{K} = \left[\begin{array}{ccccccccccccccc}
0.1 & 1.6 & 0 & & & & & & & & & & & & \\
-1.3 & -1.3 & 0 & & & & & & & & & & & & \\
0 & -0.3 & -0.7 & 0.1 & 1.6 & 0 & & & & & & & 0 & & \\
0.6 & 0 & 0 & -1.3 & -1.3 & 0 & & & & & & & & & \\
0 & 0.9 & 0 & 0 & -0.3 & -0.7 & 0.1 & 1.6 & 0 & & & & & & \\
-1.4 & 0 & -0.6 & 0.6 & 0 & 0 & -1.3 & -1.3 & 0 & & & & & & \\
 & & & 0 & 0.9 & 0 & 0 & -0.3 & -0.7 & 0.1 & 1.6 & 0 & & & \\
 & & & -1.4 & 0 & -0.6 & 0.6 & 0 & 0 & -1.3 & -1.3 & 0 & & & \\
 & & & & & & 0 & 0.9 & 0 & 0 & -0.3 & -0.7 & 0.1 & 1.6 & 0 \\
 & & & & & & -1.4 & 0 & -0.6 & 0.6 & 0 & 0 & -1.3 & -1.3 & 0 \\
 & & & & & & & & & 0 & 0.9 & 0 & 0 & -0.3 & -0.7 \\
 & & 0 & & & & & & & -1.4 & 0 & -0.6 & 0.6 & 0 & 0 \\
 & & & & & & & & & & & & 0 & 0.9 & 0 \\
 & & & & & & & & & & & & -1.4 & 0 & -0.6
\end{array}\right].$$

Appendix B

$$X(w) = \underline{B}^{\mathrm{R}}(w) =$$

$$\begin{bmatrix} -0.1825w^4 - 0.2011w^3 - 0.4937w^2 - 0.9061w - 0.6667 & 0.1950w^4 - 0.4794w^3 + 0.3289w^2 - 0.0568w - 0.8205 \\ 0.3313w^3 + 0.3537w^2 + 0.5984w + 0.6667 & -0.3539w^3 + 0.8820w^2 - 0.3219w + 0.0513 \\ 0.4259w^4 + 0.2867w^3 + 1.3463w^2 + 1.3386w + 0.9527 & -0.4550w^4 + 1.3135w^3 - 1.6692w^2 + 2.2668w - 0.7658 \end{bmatrix}.$$

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REFERENCES

[1] S. Bańka and P. Dworak, "Efficient algorithm for designing multipurpose control systems for invertible and right-invertible MIMO LTI plants", *Bull. Pol. Ac.: Tech.* 54 (4), 429–436 (2006).

[2] K.J. Latawiec, *The Power of Inverse Systems in Linear and Nonlinear Modeling and Control*, Opole University of Technology Press, Opole, Poland, 2004.

[3] R.A.J. de Vries, "Robust control through signal constraints with application to predictive control", *PhD Thesis*, Delft University of Technology, Delft, 2006.

[4] U. Borisson, "Self-tuning regulators for a class of multivariable systems", *Automatica* 15 (2), 209–215 (1979).

[5] L. Keviczky and J. Hetthessy, "Self-tuning minimum variance control of MIMO discrete time systems", *Automatic Control Theory and Applications* 5 (1), 11–17 (1977).

[6] H.N. Koivo, "A multivariable self-tuning controller", *Automatica* 16 (4), 351–366 (1980).

[7] T.J. Moir and M.J. Grimble, "Weighted minimum variance controller for nonsquare multivariable systems", *IEEE Trans. on Automatic Control* 31 (10), 976–977 (1986).

[8] K.J. Åström, *Introduction to Stochastic Control Theory*, Academic Press, New York, 1970.

[9] K.J. Åström and B. Wittenmark, "On self-tuning regulators", *Automatica* 9 (2), 185–199 (1973).

[10] R.R. Bitmead, M. Gevers, and V. Wertz, *Adaptive Optimal Control: The Thinking Man's GPC*, Prentice-Hall, Englewood Cliffs, 1990.

[11] E.F. Camacho and C. Bordons, *Model Predictive Control*, Springer-Verlag, London, 2004.

[12] D.W. Clarke and C. Mohtadi, "Properties of generalized predictive control", *Automatica* 25 (6), 859–875 (1989).

[13] M. Cychowski, *Robust Model Predictive Control*, VDM Verlag Dr. Müller AG und Co. KG, Berlin, 2009.

[14] H. Demircioğlu and P.J. Gawthrop, "Multivariable continuous-time generalized predictive control (MCGPC)", *Automatica* 28 (4), 697–713 (1992).

[15] W.P. Hunek, "Control zeros for continuous-time LTI MIMO systems and their application in theory of circuits and systems", *PhD Thesis*, Opole University of Technology, Opole, 2003, (in Polish).

[16] K.J. Latawiec, *Contributions to Advanced Control and Estimation for Linear Discrete-Time MIMO Systems*, Opole University of Technology Press, Opole, 1998.

[17] J.M. Maciejowski, *Predictive Control with Constraints*, Prentice-Hall, Harlow, 2002.

[18] C.A. Desoer and J.D. Schulman, "Zeros and poles of matrix transfer functions and their dynamical interpretation", *IEEE Trans. Circuits and Systems* 21 (1), 3–8 (1974).

[19] E.J. Davison and S.H. Wang, "Properties and calculation of transmission zeros of linear multivariable systems", *Automatica* 10 (6), 643–658 (1974).

[20] E.J. Davison and S.H. Wang, "Remark on multiple transmission zeros of a system", *Automatica* 12 (2), 195 (1976).

[21] B.A. Francis and W.M. Wonham, "The role of transmission zeros in linear multivariable regulators", *Int. J. Control* 22 (5), 657–681 (1975).

[22] A.G.J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: a survey of the algebraic, geometric and complex-variable theory", *Int. J. Control* 24 (1), 33–74 (1976).

[23] T. Kaczorek, *Control and Systems Theory*, PWN, Warsaw, 1999, (in Polish).

[24] H.H. Rosenbrock, *State-Space and Multivariable Theory*, Nelson-Wiley, New York, 1970.

[25] K.J. Latawiec, S. Bańka, and J. Tokarzewski, "Control zeros and nonminimum phase LTI MIMO systems", *Annual Reviews in Control (IFAC J.)* 24 (1), 105–112 (2000).

[26] W.P. Hunek and K.J. Latawiec, "Minimum variance control of discrete-time and continuous-time LTI MIMO systems – a new unified framework", *Control and Cybernetics* 38 (3), 609–624 (2009).

[27] K.J. Latawiec and W.P. Hunek, "Control zeros for continuous-time LTI MIMO systems", *Proc. 8th IEEE Int. Conf. on Methods and Models in Automation and Robotics* 1, 411–416 (2002).

[28] K.J. Latawiec, "Control zeros and maximum-accuracy/maximum-speed control of LTI MIMO discrete-time systems", *Control and Cybernetics* 34 (2), 453–475 (2005).

[29] M.H. Amin and M.M. Hassan, "Determination of invariant zeros and zero directions of the system S(A,B,C,E)", *Int. J. Control* 47 (4), 1011–1041 (1988).

[30] N. Karcanias and B. Kouvaritakis, "The output zeroing problem and its relationship to the invariant zero structure: a matrix pencil approach", *Int. J. Control* 30 (3), 395–415 (1979).

[31] B. Porter, "Invariant zeros and zero-directions of multivariable linear systems with slow and fast modes", *Int. J. Control* 28 (1), 81–91 (1978).

[32] C.B. Schrader and M.K. Sain, "Research on system zeros: a survey", *Int. J. Control* 50 (4), 1407–1433 (1989).

[33] U. Shaked and N. Karcanias, "The use of zeros and zero-directions in model reduction", *Int. J. Control* 23 (1), 113–135 (1976).

[34] J. Tokarzewski, *Zeros in Linear Systems: a Geometric Approach*, Warsaw University of Technology Press, Warsaw, 2002.

[35] J. Tokarzewski, "A note on some characterization of invariant zeros in singular systems and algebraic criteria of nondegeneracy", *Int. J. Applied Mathematics and Computer Science* 14 (2), 149–159 (2004).

[36] K.J. Latawiec, W.P. Hunek, R. Stanisławski, and M. Łukaniszyn, "Control zeros versus transmission zeros intriguingly revisited", *Proc. 9th IEEE Int. Conf. on Methods and Models in Automation and Robotics.* 1, 449–454 (2003).

[37] K.J. Latawiec, W.P. Hunek, and M. Łukaniszyn, "A new type of control zeros for LTI MIMO systems", *Proc. 10th IEEE Int. Conf. on Methods and Models in Automation and Robotics.* 1, 251–256 (2004).

[38] W.P. Hunek, "A robust approach to minimum variance control of LTI MIMO systems", S. Pennacchio, ed., *Emerging Technologies, Robotics and Control Systems*, Vol. 2, pp. 133–138, International Society for Advanced Research, Palermo, 2007.

[39] W.P. Hunek, "Towards robust minimum variance control of nonsquare LTI MIMO systems", *Archives of Control Sciences* 18 (1), 59–71 (2008).

[40] W.P. Hunek, "Towards robust pole-free designs of minimum variance control for LTI MIMO systems", S. Pennacchio, ed., *Recent Advances in Control Systems, Robotics and Automation*, Vol. 1, pp. 168–174, International Society for Advanced Research, Palermo, 2009.

[41] W.P. Hunek and K.J. Latawiec, "A Smith factorization approach to robust minimum variance control of nonsquare LTI MIMO systems", A. Lazinica, ed., *New Approaches in Automation and Robotics*, pp. 373–380, I-Tech Education and Publishing, Vienna, 2008.

[42] W.P. Hunek, K.J. Latawiec, and M. Łukaniszyn, "An inverse-free approach to minimum variance control of LTI MIMO systems", *Proc. 12th IEEE Int. Conf. on Methods and Models in Automation and Robotics* 1, 373–378 (2006).

[43] D. Henrion, "Reliable algorithms for polynomial matrices", *PhD Thesis*, Czech Academy of Sciences, Prague, 1998.

[44] D. Henrion, *Private Communication*, 2006.

[45] T. Kaczorek, *Private Communication*, 2005.

[46] T. Kaczorek and R. Łopatka, "Existence and computation of the set of positive solutions to polynomial matrix equations", *Int. J. Applied Mathematics and Computer Science* 10 (2), 309–320 (2000).

[47] W.P. Hunek and K.J. Latawiec, "New results on minimum phase systems", K. Malinowski and L. Rutkowski, eds., *Recent Advances in Control and Automation, Challenging Problems of Science, Control and Automation*, pp. 31–40. Academic Publishing House EXIT, Warszawa, 2008.

[48] E.J. Davison, "Some properties of minimum phase systems and 'squared-down' systems", *IEEE Trans. on Automatic Control* 28 (2), 221–222 (1983).

[49] W.P. Hunek, "A minimum-energy design of robust minimum variance control for nonsquare LTI MIMO systems", S. Pennacchio, ed., *Emerging Technologies, Robotics and Control Systems*, pp. 86–90, International Society for Advanced Research, Palermo, 2008.

[50] K.J. Latawiec, W.P. Hunek, and M. Łukaniszyn, "New optimal solvers of MVC-related linear matrix polynomial equations", *Proc. 11th IEEE Int. Conf. on Methods and Models in Automation and Robotics* 1, 333–338 (2005).

[51] W.P. Hunek, K.J. Latawiec, and R. Stanisławski, "New results in control zeros vs. transmission zeros for LTI MIMO systems", *Proc. 13th IEEE IFAC Int. Conf. on Methods and Models in Automation and Robotics* 1, 149–153 (2007).

[52] F.M. Callier and F. Kraffer, "Proper feedback compensators for a strictly proper plant by polynomial equations", *Int. J. Applied Mathematics and Computer Science* 15 (4), 493–507 (2005).

[53] J. Klamka, "Stochastic controllability of linear systems with delay in control", *Bull. Pol. Ac.: Tech.* 55 (1), 23–29 (2007).

[54] H. Martinez-Alfaro and M.A. Ruiz-Cruz, "Discrete optimal control systems design using simulated annealing", *Proc. IEEE Int. Conf. on Systems, Man and Cybernetics* 3, 2575–2580 (2003).

[55] V. Rumchev and S. Chotijah, "The minimum energy problem for positive discrete-time linear systems with fixed final state", R. Bru and S. Romero-Vivó, eds., *Positive Systems*, Vol. 389 of *Lecture Notes in Control and Information Sciences*, pp. 141–149, Springer-Verlag, Berlin, 2009.

[56] T. Kiyota and E. Kondo, "Minimum energy control for multi-input linear digital systems in z-domain", *Proc. American Control Conf.* 5, 3907–3911 (1995).