BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 59, No. 2, 2011
DOI: 10.2478/v10175-011-0024-z

Positivity and stability of fractional 2D Lyapunov systems described by the Roesser model

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Abstract. A new class of fractional 2D Lyapunov systems described by the Roesser models is introduced. Necessary and sufficient conditions for the positivity and asymptotic stability of the new class of systems are established. It is shown that the checking of the asymptotic stability of positive 2D fractional Lyapunov systems can be reduced to testing the asymptotic stability of corresponding positive standard 1D discrete-time systems. The considerations are illustrated by a numerical example.

Key words: positive system, stability, fractional 2D system, Lyapunov system, Roesser model.

1. Introduction

The most popular models of two-dimensional (2D) linear systems are the models introduced in [1–3] and [4]. These models have been extended for positive systems in [5–8]. An overview of 2D linear systems theory is given in [9–12] and some recent results in positive systems have been given in the monographs [6, 13]. The asymptotic stability of positive 2D linear systems has been investigated in [14–17]. The problem of positivity and stabilization of 2D linear systems by state-feedbacks has been considered in [18].

Mathematical fundamentals of fractional calculus are given in the monographs [19–23]. The notion of fractional 2D linear systems has been introduced by Kaczorek in [24] and has been extended in [25, 26]. The problem of positivity and stabilization of 2D fractional systems by state-feedbacks has been considered in [27, 28].

Controllability and observability of Lyapunov systems have been investigated in [29]. Positive 1D Lyapunov systems have been considered in [30–34] and positive 2D Lyapunov systems have been analysed in [35].

In [36] a new fractional Lyapunov model has been introduced and has been extended in [37]. The positivity, stability, observability, reachability and controlability to zero problems for this model have been formulated and solved.

In this paper a new class of 2D fractional Lyapunov systems will be introduced and necessary and sufficient conditions for the positivity and asymptotic stability will be established.

To the best knowledge of the author 2D fractional Lypunov systems, its positivity and stability has not been considered yet.

2. Preliminaries

Let $\mathbb{R}_+^{n\times m}$ be the set of $n\times m$ matrices with all nonnegative elements and $\mathbb{R}_+^n := \mathbb{R}_+^{n\times 1}$. The set of nonnegative integers will be denoted by $\mathbb{Z}_+$ and the $n\times n$ identity matrix will be denoted by $\mathbb{I}_n$. A matrix $A=[a_{ij}]\in\mathbb{R}_+^{n\times m}$ will be called strictly positive and denoted by $A>0$ if $a_{ij}>0$ for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$.

A square real matrix $A=\left[a_{ij}\right]$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij}\geq 0$ for $i\neq j$.

Definition 1. [38, p. 80] The Kronecker product $A\otimes B$ of matrices $A=\left[a_{ij}\right]\in\mathbb{R}^{m\times n}$ and $B\in\mathbb{R}^{p\times q}$ is the block matrix

$$A\otimes B=\left[a_{ij}B\right]_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}}\in\mathbb{R}^{mp\times nq}. \tag{1}$$

Lemma 1. [38, p. 82] The equation

$$AXB=C, \tag{2}$$

where $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{q\times p}$, $C\in\mathbb{R}^{m\times p}$ and $X\in\mathbb{R}^{n\times q}$ is equivalent to the following one

$$\left(A\otimes B^T\right)x=c, \tag{3}$$

where

$$\begin{aligned} x &:= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T, \\ c &:= \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}^T \end{aligned} \tag{4}$$

and x_i and c_i are the i-th rows of the matrices X and C, respectively.

Lemma 2. [38, p. 385] If $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of the matrix $A\in\mathbb{R}^{n\times n}$ and $\mu_1,\mu_2,\ldots,\mu_n$ are the eigenvalues of the matrix $B\in\mathbb{R}^{n\times n}$, then $\lambda_i+\mu_j$ for $i,j=1,2,\ldots,n$ are the eigenvalues of the matrix

$$\bar{A}=A\otimes\mathbb{I}_n+\mathbb{I}_n\otimes B^T.$$

The following notions of fractional differences of 2D horizontal and vertical matrix functions will be introduced.

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Definition 2. The α-order fractional difference of an 2D horizontal matrix function $X_{ij}^h \in \mathbb{R}^{n_1 \times N}$, $i,j \in \mathbb{Z}_+$ is defined by the formula

$$\Delta^\alpha X_{ij}^h = \sum_{k=0}^{i} c_\alpha(k) X_{i-k,j}^h, \tag{4a}$$

where $\alpha \in \mathbb{R}$, $\quad n-1 < \alpha < n \in \mathbb{N} = \{1,2,\dots\}$ and

$$c_\alpha(k) = \begin{cases} 1 & \text{for } k=0 \\ (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for } k>0 \end{cases}. \tag{4b}$$

Definition 3. The β-order fractional difference of an 2D vertical matrix function $X_{ij} \in \mathbb{R}^{n_2 \times N}$, $i,j \in \mathbb{Z}_+$ is defined by the formula

$$\Delta^\beta X_{ij}^v = \sum_{l=0}^{j} c_\beta(l) X_{i,j-l}^v, \tag{5a}$$

where $\beta \in \mathbb{R}$, $\quad n-1 < \beta < n \in \mathbb{N}$ and

$$c_\beta(l) = \begin{cases} 1 & \text{for } l=0 \\ (-1)^l \binom{\beta}{l} = (-1)^l \frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} & \text{for } l>0 \end{cases}. \tag{5b}$$

Definitions 2 and 3 are a generalization of fractional partial differences of 2D discrete functions given in [23] and [28].

Lemma 3. [28] If $n-1 < \alpha < n \in \mathbb{N} = \{1,2,\dots\}$ $(n-1 < \beta < n)$ then

$$\sum_{k=0}^{\infty} c_\alpha(k) = 0 \quad \left(\sum_{k=0}^{\infty} c_\beta(k) = 0 \right). \tag{6}$$

3. Fractional 2D Lyapunov system and its solution

Consider the fractional 2D linear Lyapunov system described by the state equations

$$\begin{bmatrix} \Delta^\alpha X_{i+1,j}^h \\ \Delta^\beta X_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} + \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U_{ij}, \tag{7a}$$

$$Y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} + DU_{ij} \quad i,j \in \mathbb{Z}_+, \tag{7b}$$

where $X_{ij}^h \in \mathbb{R}^{n_1 \times N}$, $X_{ij}^v \in \mathbb{R}^{n_2 \times N}$ are horizontal and vertical state matrix at the point (i,j) respectively, $U_{ij} \in \mathbb{R}^{m \times N}$ is input matrix, $Y_{ij} \in \mathbb{R}^{p \times N}$ is output matrix at the point (i,j) and $A_{kl}^r \in \mathbb{R}^{n_k \times n_l}$ for $k,l = 1,2$ and $r = 0,1$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $B_2 \in \mathbb{R}^{n_2 \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$, $C_2 \in \mathbb{R}^{p \times n_2}$, $D \in \mathbb{R}^{p \times m}$ and $N = n_1 + n_2$.

Using Definition 2 and Definition 3 we may write the Eq. (7a) in the form

$$\begin{bmatrix} X_{i+1,j}^h \\ X_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}^0 & A_{12}^0 \\ A_{21}^0 & \overline{A}_{22}^0 \end{bmatrix} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} + \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) X_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_\beta(l) X_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U_{ij} \tag{8}$$

where $\overline{A}_{11}^0 = A_{11}^0 + \alpha \mathbb{I}_{n_1}$, $\overline{A}_{22}^0 = A_{22}^0 + \beta \mathbb{I}_{n_2}$.

The boundary conditions for the Eqs. (7a) and (7b) are given in the form

$$X_{0j}^h \text{ for } j \in \mathbb{Z}_+, \qquad X_{i0}^v \text{ for } i \in \mathbb{Z}_+. \tag{9}$$

Lemma 4. The 2D Lyapunov system (7) can be transformed to the equivalent fractional 2D Nm-input and Np-output system described by the Roesser model in the form [28]

$$\begin{bmatrix} \overline{x}_{i+1,j}^h \\ \overline{x}_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} \overline{x}_{ij}^h \\ \overline{x}_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) \overline{x}_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_\beta(l) \overline{x}_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} \overline{u}_{ij}, \tag{10a}$$

$$\overline{y}_{ij} = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix} \begin{bmatrix} \overline{x}_{ij}^h \\ \overline{x}_{ij}^v \end{bmatrix} + \overline{D}\overline{u}_{ij}, \qquad i,j \in \mathbb{Z}_+, \tag{10b}$$

where

$$\begin{aligned} \overline{x}_{ij}^h &= \begin{bmatrix} {}_1X_{ij}^h & {}_2X_{ij}^h & \dots & {}_{n_1}X_{ij}^h \end{bmatrix}^T \in \mathbb{R}^{N \cdot n_1}, \\ \overline{x}_{ij}^v &= \begin{bmatrix} {}_1X_{ij}^v & {}_2X_{ij}^v & \dots & {}_{n_2}X_{ij}^v \end{bmatrix}^T \in \mathbb{R}^{N \cdot n_2}, \\ \overline{u}_{ij} &= \begin{bmatrix} {}_1U_{ij} & {}_2U_{ij} & \dots & {}_mU_{ij} \end{bmatrix}^T \in \mathbb{R}^{N \cdot m}, \\ \overline{y}_{ij} &= \begin{bmatrix} {}_1Y_{ij} & {}_2Y_{ij} & \dots & {}_pY_{ij} \end{bmatrix}^T \in \mathbb{R}^{N \cdot p} \end{aligned} \tag{11}$$

and ${}_kX_{ij}^h$, ${}_kX_{ij}^v$, ${}_kU_{ij}$, ${}_kY_{ij}$ denote the k–th rows of the matrices X_{ij}^h, X_{ij}^v, U_{ij}, Y_{ij}, respectively,

$$
\begin{aligned}
&\overline{A}_{11} = \overline{A}_{11}^{0} \otimes \mathbb{I}_N + \mathbb{I}_{n_1} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T \in \\
&\qquad \in \mathbb{R}^{(N\cdot n_1)\times(N\cdot n_1)}, \\
&\overline{A}_{12} = A_{12}^0 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot n_1)\times(N\cdot n_2)}, \\
&\overline{A}_{21} = A_{21}^0 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot n_2)\times(N\cdot n_1)}, \\
&\overline{A}_{22} = \overline{A}_{22}^{0} \otimes \mathbb{I}_N + \mathbb{I}_{n_2} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T \in \\
&\qquad \in \mathbb{R}^{(N\cdot n_2)\times(N\cdot n_2)}, \\
&\overline{B}_1 = B_1 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot n_1)\times(N\cdot m)}, \\
&\overline{B}_2 = B_2 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot n_2)\times(N\cdot m)}, \\
&\overline{C}_1 = C_1 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot p)\times(N\cdot n_1)}, \\
&\overline{C}_2 = C_2 \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot p)\times(N\cdot n_2)}, \\
&\overline{D} = D \otimes \mathbb{I}_N \in \mathbb{R}^{(N\cdot p)\times(N\cdot m)}.
\end{aligned}
\tag{12}
$$

Proof. Using Lemma 1 for the Eq. (8) we obtain immediately (10).

The boundary conditions for the Eq. (10) are given in the form

$$
\begin{aligned}
\overline{x}_{0j}^h &= \begin{bmatrix} {}_1X_{0j}^h & {}_2X_{0j}^h & \dots & {}_{n_1}X_{0j}^h \end{bmatrix}^T \quad \text{for } j \in \mathbb{Z}_+, \\
\overline{x}_{ij}^v &= \begin{bmatrix} {}_1X_{i0}^v & {}_2X_{i0}^v & \dots & {}_{n_2}X_{i0}^v \end{bmatrix}^T \quad \text{for } i \in \mathbb{Z}_+.
\end{aligned}
\tag{13}
$$

Theorem 1. The solution of Eq. (10) with boundary conditions (13) is given by

$$
\begin{aligned}
\begin{bmatrix} \overline{x}_{ij}^h \\ \overline{x}_{ij}^v \end{bmatrix} &= \sum_{p=0}^{i} \overline{T}_{i-p,j} \begin{bmatrix} 0 \\ \overline{x}_{p0}^v \end{bmatrix} + \sum_{q=0}^{j} \overline{T}_{i,j-q} \begin{bmatrix} \overline{x}_{0q}^h \\ 0 \end{bmatrix} \\
&+ \sum_{p=0}^{i}\sum_{q=0}^{j} \left(\overline{T}_{i-p-1,j-q}\overline{B}^{10} + \overline{T}_{i-p,j-q-1}\overline{B}^{01} \right) \overline{u}_{pq}
\end{aligned}
\tag{14a}
$$

where

$$
\overline{B}^{10} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}, \qquad \overline{B}^{01} = \begin{bmatrix} 0 \\ \overline{B}_2 \end{bmatrix}
\tag{14b}
$$

and the transition matrices $\overline{T}_{pq} \in \mathbb{R}^{N^2\times N^2}$ are defined by the formula

$$
\overline{T}_{pq} = \begin{cases} \mathbb{I}_{N^2} & \text{for } p=0,\ q=0 \\ \overline{\mathbb{T}}_{pq} & \text{for } p+q>0\ (p,q \in \mathbb{Z}_+) \\ 0 \text{ (zero matrix)} & \text{for } p<0 \text{ and/or } q<0 \end{cases}
\tag{14c}
$$

where

$$
\begin{aligned}
\overline{\mathbb{T}}_{pq} &= \overline{T}_{10}\overline{T}_{p-1,q} - \sum_{k=2}^{p} \begin{bmatrix} c_\alpha(k)\mathbb{I}_{(N\cdot n_1)} & 0 \\ 0 & 0 \end{bmatrix} \overline{T}_{p-k,q} + \\
&+ \overline{T}_{01}\overline{T}_{p,q-1} - \sum_{l=2}^{q} \begin{bmatrix} 0 & 0 \\ 0 & c_\beta(l)\mathbb{I}_{(N\cdot n_2)} \end{bmatrix} \overline{T}_{p,q-l}
\end{aligned}
\tag{14d}
$$

and

$$
\overline{T}_{10} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{bmatrix}, \quad \overline{T}_{01} = \begin{bmatrix} 0 & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}
\tag{14e}
$$

Proof. The theorem results directly from the state-space equation solution of the fractional 2D linear system described by the Roesser model (10), see [28].

4. Positivity of the fractional 2D Lyapunov system

Definition 4. The system (7) is called the (internally) positive fractional 2D Lyapunov system if $X_{ij}^h \in \mathbb{R}_+^{n_1\times N}$, $X_{ij}^v \in \mathbb{R}_+^{n_2\times N}$ and $Y_{ij} \in \mathbb{R}_+^{p\times N}$, $i,j \in \mathbb{Z}_+$ for any nonnegative boundary conditions $X_{0j}^h \in \mathbb{R}_+^{n_1\times N}$, $j \in \mathbb{Z}_+$ and $X_{i0}^v \in \mathbb{R}_+^{n_2\times N}$, $i \in \mathbb{Z}_+$ and all input sequences $U_{ij} \in \mathbb{R}_+^{m\times N}$, $i,j \in \mathbb{Z}_+$.

Theorem 2. The fractional 2D Lyapunov system (8) for $\alpha,\beta \in \mathbb{R}$, $0<\alpha\le 1$, $0<\beta\le 1$ is positive if and only if

$$
A_{kk}^l = \left[{}_{ij}a_{kk}^l\right]_{i,j=1,\dots,n_k} \quad \text{for} \quad k=1,2;\ l=0,1
\tag{15a}
$$

are Metzler matrices satisfying

$$
\begin{aligned}
&{}_{ii}a_{11}^0 + \alpha + {}_{jj}a_{11}^1 \ge 0 \quad \text{for} \quad i,j=1,\dots,n_1 \\
&{}_{ii}a_{11}^0 + \alpha + {}_{jj}a_{22}^1 \ge 0 \quad \text{for} \quad i=1,\dots,n_1,\ j=1,\dots,n_2 \\
&{}_{ii}a_{22}^0 + \beta + {}_{jj}a_{11}^1 \ge 0 \quad \text{for} \quad i=1,\dots,n_2,\ j=1,\dots,n_1 \\
&{}_{ii}a_{22}^0 + \beta + {}_{jj}a_{22}^1 \ge 0 \quad \text{for} \quad i,j=1,\dots,n_2
\end{aligned}
\tag{15b}
$$

and

$$
\begin{aligned}
&A_{kl}^r \in \mathbb{R}_+^{n_k\times n_l} \quad \text{for} \quad k,l=1,2;\ k\ne l;\ r=0,1; \\
&B_k \in \mathbb{R}_+^{n_k\times m}, \quad C_k \in \mathbb{R}_+^{p\times n_k} \quad \text{for} \quad k=1,2; \\
&D \in \mathbb{R}_+^{p\times m}.
\end{aligned}
\tag{15c}
$$

Proof. The fractional 2D Lyapunov system (8) is positive if and only if the equivalent fractional 2D system (10) is positive. By the theorem of the positivity of fractional 2D linear system described by the Roesser model [28] we have

$$
\begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} = \mathbb{R}_+^{N^2\times N^2}, \quad \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = \mathbb{R}_+^{N^2\times (N\cdot m)},
$$

$$
\begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix} = \mathbb{R}_+^{(N\cdot p)\times N^2}, \quad \overline{D} = \mathbb{R}_+^{(N\cdot p)\times (N\cdot m)}.
$$

Using (12) we obtain (15).

5. Asymptotic stability of fractional 2D Lyapunov systems

Definition 5. The positive fractional Lyapunov system (7) is called asymptotically stable if for any bounded boundary conditions $X_{0j}^h \in \mathbb{R}_+^{n_1\times N}$ for $j \in \mathbb{Z}_+$, $X_{i0}^v \in \mathbb{R}_+^{n_2\times N}$ for $i \in \mathbb{Z}_+$ and zero inputs $U_{ij} = 0$ for $i,j \in \mathbb{Z}_+$ we have

$$
\lim_{i,j\to\infty} \begin{bmatrix} X_{ij}^h \\ X_{ij}^v \end{bmatrix} = 0.
\tag{16}
$$

Theorem 3. The positive fractional 2D Lyapunov system (7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. the positive 1D system
$$x_{i+1} = \begin{bmatrix} \tilde{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \tilde{A}_{22} \end{bmatrix} x_i \tag{17}$$
where
$$\tilde{A}_{kk} = (A^0_{kk} + \mathbb{I}_{n_k}) \otimes \mathbb{I}_N + \mathbb{I}_{n_k} \otimes \begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix}^T, \quad \text{for } k = 1, 2 \tag{18}$$
and $\overline{A}_{12}$, $\overline{A}_{21}$ are given by (12) is asymptotically stable,
2. $$\left|\lambda_i + \mu_j\right| < 1 \quad \text{for } i, j = 1, 2 \ldots, N \tag{19}$$
where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the eigenvalues of the matrix
$$\begin{bmatrix} A^0_{11} + \mathbb{I}_{n_1} & A^0_{12} \\ A^0_{21} & A^0_{22} + \mathbb{I}_{n_2} \end{bmatrix}$$
and $\mu_1, \mu_2, \ldots, \mu_N$ are the eigenvalues of the matrix
$$\begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix}$$
3. there exists a strictly positive matrix $\Lambda \in \mathbb{R}^{N \times N}_+$ such that
$$\begin{bmatrix} A^0_{11} & A^0_{12} \\ A^0_{21} & A^0_{22} \end{bmatrix} \Lambda + \Lambda \begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix} < 0. \tag{20}$$

Proof. By Lemma 4, the asymptotic stability of the fractional 2D Lyapunov system is equivalent to the asymptotic stability of the system (10). Note, that this system is the system with delays. The number of delays increases for $i, j \to \infty$. In [39, 40] it was shown that the asymptotic stability of the positive discrete-time linear system with delays is independent of the number and values of the delays and depends only on the sum of the state matrices. Therefore, the asymptotic stability of the positive 2D fractional system (10) is equivalent to the asymptotic stability of the positive 1D standard system with the matrix
$$\begin{bmatrix} \overline{A}_{11} - \sum\limits_{k=2}^{\infty} c_\alpha(k) \mathbb{I}_{(N \cdot n_1)} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} - \sum\limits_{l=2}^{\infty} c_\beta(l) \mathbb{I}_{(N \cdot n_2)} \end{bmatrix}. \tag{21}$$
Using Lemma 3 and from (4b), (5b) we obtain
$$\sum_{k=2}^{\infty} c_\alpha(k) = \alpha - 1 \quad \text{and} \quad \sum_{k=2}^{\infty} c_\beta(k) = \beta - 1. \tag{22}$$
Substitution of (22) into (21) yields (17).

It is well-known that 1D discrete-time system (17) is asymptotically stable if and only if all eigenvalues of the system matrix have moduli less than one. Using Lemma 2 we obtain (19).

In [41], it was shown that the positive 1D system (17) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}^N_+$ such that
$$\begin{bmatrix} \tilde{A}_{11} - \mathbb{I}_{(N \cdot n_1)} & \overline{A}_{12} \\ \overline{A}_{21} & \tilde{A}_{22} - \mathbb{I}_{(N \cdot n_2)} \end{bmatrix} \lambda < 0$$
Applying Lemma 1 and from (18) we obtain (20).

Let us denote
$$\begin{bmatrix} A^r_{11} & A^r_{12} \\ A^r_{21} & A^r_{22} \end{bmatrix} = \left[a^r_{kl}\right]_{k,l=1,\ldots,N} \quad \text{for } r = 0, 1 \tag{23}$$
and
$$\Lambda = \left[\lambda_{kl}\right]_{k,l=1,\ldots,N}. \tag{24}$$

Theorem 4. The positive fractional 2D Lyapunov system (7) is asymptotically stable only if
$$\begin{gathered} a^0_{kk} + a^1_{ll} \in \left[-\alpha, 0\right) \\ \text{for } k = 1, 2, \ldots, n_1;\ l = 1, 2, \ldots, N; \\ a^0_{kk} + a^1_{ll} \in \left[-\beta, 0\right) \\ \text{for } k = n_1 + 1, n_1 + 2, \ldots, N;\ l = 1, 2, \ldots, N \end{gathered} \tag{25}$$

Proof. The inequality (20) can be written in the form
$$\begin{aligned} \sum_{j=1}^{N} a^0_{kj} \lambda_{jl} + \sum_{j=1}^{N} \lambda_{kj} a^1_{jl} = \sum_{\substack{j=1 \\ j \neq k}}^{N} a^0_{kj} \lambda_{jl} + \sum_{\substack{j=1 \\ j \neq l}}^{N} \lambda_{kj} a^1_{jl} + \\ + (a^0_{kk} + a^1_{ll}) \lambda_{kl} < 0 \quad \text{for } k, l = 0, 1, \ldots, N. \end{aligned} \tag{26}$$
By Theorem 2 we have that the inequality (26) can be satisfied only if the conditions of Theorem 4 are hold.

Example 1. Consider the fractional 2D Lyapunov system (7) for $\alpha = 0.8$, $\beta = 0.5$ with matrices
$$\begin{gathered} \begin{bmatrix} A^0_{11} & A^0_{12} \\ A^0_{21} & A^0_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0.3 & -0.2 \end{bmatrix}, \\ \begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \quad B_1 = B_2 = 0. \end{gathered} \tag{27}$$

By Theorem 2 this system is positive since the matrices A^r_{11}, A^r_{22} for $r = 0, 1$ are Metzler matrices satisfying
$$\begin{gathered} {}_{11}a^0_{11} + \alpha + {}_{11}a^1_{11} = 0.1 > 0, \\ {}_{11}a^0_{11} + \alpha + {}_{11}a^1_{22} = 0.2 > 0, \\ {}_{11}a^0_{22} + \beta + {}_{11}a^1_{11} = 0.1 > 0, \\ {}_{11}a^0_{22} + \beta + {}_{11}a^1_{22} = 0.2 > 0 \end{gathered}$$
and the remaining matrices of the system have all nonnegative entries.

Applying Theorem 3 we obtain the following results.

1. The 1D system with the matrix

$$\begin{bmatrix} \tilde{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.6 & 0 \\ 0 & 0.3 & 0.1 & 0.7 \end{bmatrix}$$

is asymptotically stable since this matrix has eigenvalues with moduli $|z_1| = 0.7$, $|z_2| = 0.4$, $|z_3| = 0.6$, $|z_4| = 0.3$.

2. Taking into account that the matrix

$$\begin{bmatrix} A_{11}^0 + \mathbb{I}_{n_1} & A_{12}^0 \\ A_{21}^0 & A_{22}^0 + \mathbb{I}_{n_2} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0.3 & 0.8 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 0.8$, $\lambda_2 = 0.5$ and the matrix

$$\begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix}$$

has eigenvalues $\mu_1 = -0.2$, $\mu_2 = -0.1$ we obtain

$$|\lambda_1 + \mu_1| = 0.6 < 1, \qquad |\lambda_1 + \mu_2| = 0.7 < 1,$$
$$|\lambda_2 + \mu_1| = 0.3 < 1, \qquad |\lambda_2 + \mu_2| = 0.4 < 1.$$

3. There exists strictly positive matrix

$$\Lambda = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

such that the inequality

$$\begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} \Lambda + \Lambda \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \\ = \begin{bmatrix} -0.7 & -0.5 \\ -0.1 & -0.2 \end{bmatrix} < 0$$

holds.

Therefore, by Theorem 3 the fractional 2D Lyapunov system (7) with matrices (27) is asymptotically stable.

Note that the conditions of Theorem 4 are also met since

$$a_{11}^0 + a_{11}^1 = -0.7 \in [-0.8, 0),$$

$$a_{11}^0 + a_{22}^1 = -0.6 \in [-0.8, 0),$$

$$a_{22}^0 + a_{11}^1 = -0.4 \in [-0.5, 0),$$

$$a_{22}^0 + a_{22}^1 = -0.3 \in [-0.5, 0).$$

6. Concluding remarks

The notion of a positive 2D fractional Lyapunov system described by the Roesser model has been introduced. For this model necessary and sufficient conditions for positivity and asymptotic stability have been established. It has been shown that checking the asymptotic stability of positive fractional 2D Lyapunov systems can be reduced to testing the stability of corresponding positive standard 1D discrete-time linear systems. The considerations have been illustrated by numerical example.

An open problem is extension of the considerations for 2D Lyapunov systems described by the models with structure similar to the Kurek model [4].

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