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Observability of linear q-difference fractional-order systems with finite initial memory

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Abstract. In this paper we define a class of linear q-difference fractional-order systems with finite memory. For such systems we state definitions of indistinguishability and observability by using the concept of extended initial conditions. We prove the formula for the solution and the rank condition for observability.

Key words: linear q-difference fractional-order systems, l-memory initial value problem, l-event, observability in s-steps.

1. Introduction

The fractional calculus in continuous case includes different notion of derivatives, e.g. Riemann-Liouville, Grünwald-Letnikov, Caputo and generalized function approach [1, 2]. In modeling the real phenomena authors emphatically use generalizations of n-th order differences to their fractional forms and consider the state-space equations of control systems in discrete-time, e.g. [3, 4]. As the unification of both cases with classical tools one can consider systems on time scales [5, 6]. The theory of q-difference linear control systems is developed separately as a special kind of systems on time scales, see e.g. [7–9].

In the generalization of classical discrete differences to fractional-order there is convenient to take finite summation, see [3, 4, 10, 11]. What seems to be one of the greatest phenomena in using fractional derivatives/differences in systems modeling real behaviors is the initialization of the process. In fact the initial value problem is an important task in daily applications. Recently we can find papers dealing with the problem how to impose initial conditions, e.g. [12–14]. We propose the condition on the observability property of the special kind of systems with extended initial conditions. It is known that the observability problem is under investigation for the continuous case as well as for the discrete case, also for classical notations of differences, see [15].

In this paper we deal with a q-fractional difference system with the initialization given by the additional function φ that vanishes at a time interval with infinitely many points. In that way we get only the finite number of values of an initializing function φ that can be nonzero. We call such set by l-memory. It could be treated as the special kind of initial conditions.

The solution of the l-memory initial value problem is constructed. We use a definition of an undistinguishability relation and observability in s-steps, similarly as it is proposed in [16]. In that way we state the problem in the classical way, using the rank of the defined observability matrix.

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2. Fractional q-difference systems

Let $q \in (0,1)$ and α be any nonzero rational number. Firstly, we need the following q-analogue of $n!$, introduced in [8]:

$$[n]! = \begin{cases} 1, & \text{if } n = 0, \\ [n]\,[n-1]\cdots[1]\,, & \text{if } n = 1, 2, \ldots \end{cases}$$

Hence $[n+1]! = [n]![n+1]$ for each $n \in \mathbb{N}$.

Following the notations in [8], we write $[\alpha] = \dfrac{1-q^\alpha}{1-q}$ and for generalization of the q-binomial coefficients

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} \alpha \\ j \end{bmatrix} = \frac{[\alpha][\alpha-1]\cdots[\alpha-j+1]}{[j]!}, \; j \in \mathbb{N}\,.$$

Note that

(i) $[1] = 1$ but $[n+1] = 1 + q + \cdots + q^n$ and $\lim\limits_{n\to+\infty}[n] = \dfrac{1}{1-q}$.

(ii) For $n \in \mathbb{N}$: $\lim\limits_{q\to 1}[n]! = n!$.

(iii) $\begin{bmatrix} \alpha \\ 1 \end{bmatrix} = [\alpha], \quad \begin{bmatrix} \alpha \\ 2 \end{bmatrix} = \dfrac{(1-q^{\alpha-1})(1-q^\alpha)}{(1-q^2)(1-q)}.$

Using the definition of the fractional quantum derivative (see [9]), we introduce a q-difference of the fractional order as follows.

Definition 2.1. A q-difference of fractional order α of a function $x(\cdot) : \mathbb{R} \to \mathbb{R}$ at $t \neq 0$ is defined by

$$\Delta_q^\alpha x(t) := t^{-\alpha} \frac{\sum_{j=0}^{\infty} \begin{bmatrix} \alpha \\ j \end{bmatrix} (-1)^j q^{\frac{j(j+1)}{2}} q^{-j\alpha}}{(1-q)^\alpha} x(q^j t)\,.$$

Let us denote $b_j = \begin{bmatrix} \alpha \\ j \end{bmatrix} (-1)^j q^{\frac{j(j+1)}{2}} q^{-j\alpha}$. Then

$$\Delta_q^\alpha x(t) = t^{-\alpha} \sum_{j=0}^{\infty} \frac{b_j}{(1-q)^\alpha} x(q^j t)\,. \tag{1}$$

Definition 2.2. Let $l \in \mathbb{N} \cup \{0\}$ and $t_0 > 0$. Then $\Omega_l(t_0) := \{t_0, qt_0, \ldots, q^l t_0\}$.

Let $a > 0$. Then by $u_a : \mathbb{R} \to \{0, 1\}$ we denote the Heaviside step function such that $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$.

Proposition 2.3. Let $a > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}^n$ be any function and $x(t) = \varphi(t) u_a(t)$. Then, for $\Delta_q^\alpha x(t) = 0$ for $t < a$ and

$$\Delta_q^\alpha x(t) = t^{-\alpha} \sum_{j=0}^{N(t,a)} \frac{b_j}{(1-q)^\alpha} x(q^j t)\,, \tag{2}$$

for $t \geq a$, where $N(t, a)$ is the integer part of the value $\dfrac{\ln a - \ln t}{\ln q}$.

Definition 2.4. Let $l \in \mathbb{N} \cup \{0\}$, $t_0 > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}^n$. The vector

$$\mathcal{M}(l, t_0, \varphi) := \begin{bmatrix} \varphi(t_0) \\ \varphi(qt_0) \\ \vdots \\ \varphi(q^l t_0) \end{bmatrix}$$

of values of the function φ on $\Omega_l(t_0)$, is called a finite l-memory at t_0.

Remark 2.5. Let $l \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N} \cup \{0\}, t_0 > 0$, $\varphi : \mathbb{R} \to \mathbb{R}^n$. Then,

(i) $\mathcal{M}(l, t_0, \varphi) \in \mathbb{R}^{n+nl}$;

(ii) if $l_1, l_2 \in \mathbb{N} \cup \{0\}$, $l_2 \geq l_1$, then $\Omega_{l_1}(t_0) \subset \Omega_{l_2}(t_0)$ and

$$\left[I_{nl_1}, \mathbf{0}_{nl_1 \times n(l_2 - l_1)}\right] \mathcal{M}(l_2, t_0, \varphi) = \mathcal{M}(l_1, t_0, \varphi)\,,$$

where $\mathbf{0}_{nl_1 \times n(l_2 - l_1)}$ denotes the zero matrix of dimension $nl_1 \times n(l_2 - l_1)$, and I_{nl_1} is the identity matrix of degree nl_1.

Definition 2.6. Let $l \in \mathbb{N} \cup \{0\}$ and $t_0 > 0$, $a = q^l t_0$, $\varphi : \mathbb{R} \to \mathbb{R}^n$. A linear q-difference fractional-order system with l-memory, denoted by $\Sigma_{(\alpha,q,l)}$, is a system given by the following set of equations:

$$\Delta_q^\alpha x(t) = Ax(qt), \quad t > t_0, \tag{3}$$

$$x(t) = (\varphi u_a)(t), \quad t \leq t_0, \tag{4}$$

$$y(t) = Cx(t), \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{p \times n}$ are constant matrices. By $\gamma(t, t_0, M(l, t_0, \varphi))$ we denote the solution of (3)–(4) starting at t_0 and evaluated at time $t = \dfrac{t_0}{q^k}$, $k > 0$, of l-memory initial value problem corresponding to the values of the function φ.

From Eqs. (2) and (3) follows

$$x\left(\frac{t_0}{q}\right) = \left(\left(\frac{t_0(1-q)}{q}\right)^\alpha A - b_1 I_n\right) x(t_0) - \\ - \sum_{j=1}^{l} b_{j+1} x(q^j t_0) \tag{6}$$

and more generally

$$x\left(\frac{t_0}{q^{k+1}}\right) = \left(\left(\frac{t_0(1-q)}{q^{k+1}}\right)^\alpha A - b_1 I_n\right) x\left(\frac{t_0}{q^k}\right) - \\ - \sum_{j=1}^{k+l} b_{j+1} x\left(q^{j-k} t_0\right).$$

Let us set $G(k, t_0) = \left(\dfrac{t_0(1-q)}{q^{k+1}}\right)^\alpha A - b_1 I_n$, and $A_0 = \mathbf{0}_n$, while for $j > 0$: $A_j = -b_{j+1} I_n$, where $\mathbf{0}_n$ is the zero matrix of the dimension $n \times n$. Note that $G(0, t_0) = \left(\dfrac{t_0(1-q)}{q}\right)^\alpha A - b_1 I_n$. This leads to

$$x\left(\frac{t_0}{q^{k+1}}\right) = G(k, t_0) x\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k+l} A_j x\left(\frac{t_0}{q^{k-j}}\right). \tag{7}$$

Let us construct, using an idea given in [3], the sequence of matrices from $\mathbb{R}^{n \times (nl+n)}$ as follows.

$$\widetilde{\Phi}(0, t_0) = [I_n,\ \mathbf{0}_n,\ \ldots\ \mathbf{0}_n]\,,$$

$$\widetilde{\Phi}(1, t_0) = [G(0, t_0),\ A_1,\ \ldots\ A_l]\,,$$

$$\widetilde{\Phi}(2, t_0) = G(1, t_0)\widetilde{\Phi}(1, t_0) + [A_1,\ \ldots,\ A_{l+1}]$$

and for $k + 1 \geq 3$:

$$\widetilde{\Phi}(k+1, t_0) = G(k, t_0)\widetilde{\Phi}(k, t_0) + \\ + \sum_{j=1}^{k-1} A_j \widetilde{\Phi}(k - j, t_0) + [A_k, A_{k+1}, \ldots, A_{k+l}]\,.$$

Moreover, let $\Phi(k, t_0) := \widetilde{\Phi}(k, t_0) \begin{bmatrix} I_n \\ \mathbf{0}_{n \times (nl)} \end{bmatrix}$. Particularly

$$\Phi(0, t_0) = I_n,$$

$$\Phi(1, t_0) = G(0, t_0)$$

and

$$\Phi(2, t_0) = G(1, t_0)G(0, t_0) + A_1.$$

Theorem 2.7. Let $l \in \mathbb{N} \cup \{0\}$ and $t_0 > 0$, $a = q^l t_0$, $\varphi : \mathbb{R} \to \mathbb{R}^n$. The solution of the l-memory initial value problem is given by:

$$\gamma(t, t_0, M(l, t_0, \varphi)) = \widetilde{\Phi}(k, t_0)\widetilde{x}(t_0) \\ \text{for} \quad t = \frac{t_0}{q^k}, \quad k > 0, \tag{8}$$

where $\widetilde{x}(t_0) = M(l, t_0, \varphi)$.

Proof. Let $t = \frac{t_0}{q^k}$. Note that for $k = 0$ holds

$$\widetilde{\Phi}(0, t_0)\widetilde{x}(t_0) = I_n x(t_0) = x(t_0).$$

Now let us assume that the formula (8) holds for all $t = \frac{t_0}{q^j}$, $j \leq k$, $k > 0$. Consider Eq. (7) and write the formula (8) for $t = \frac{t_0}{q^{k+1}}$. Hence

$$\gamma\left(t, t_0, M(l, t_0, \varphi)\right) = G(k, t_0)x\left(\frac{t_0}{q^k}\right) + \sum_{j=1}^{k+l} A_j x\left(\frac{t_0}{q^{k-j}}\right).$$

The inductive assumption implies that

$$\gamma\left(t, t_0, M(l, t_0, \varphi)\right) = G(k, t_0)\widetilde{\Phi}\left(\frac{t_0}{q^k}, t_0\right)\widetilde{x}(t_0) +$$

$$+ A_1 x\left(\frac{t_0}{q^{k-1}}\right) + \cdots + A_{k-1}x\left(\frac{t_0}{q}\right) +$$

$$+ A_k\varphi(t_0) + A_{k+1}\varphi\left(qt_0\right) + \cdots + A_{k+l}\varphi\left(q^l t_0\right).$$

After using again inductive assumption for each of $x\left(\frac{t_0}{q^j}\right)$, $j = 1, \ldots, k-1$:

$$x\left(\frac{t_0}{q^j}\right) = \gamma\left(\frac{t_0}{q^j}, t_0, M(l, t_0, \varphi)\right) = \widetilde{\Phi}\left(\frac{t_0}{q^j}, t_0\right)\widetilde{x}(t_0)$$

and the fact

$$A_1 x\left(\frac{t_0}{q^{k-1}}\right) + \cdots + A_{k-1}x\left(\frac{t_0}{q}\right) =$$

$$= \sum_{j=1}^{k-1} A_j\widetilde{\Phi}\left(\frac{t_0}{q^{k-j}}, t_0\right)\widetilde{x}(t_0),$$

finally we get

$$\gamma\left(t, t_0, M(l, t_0, \varphi)\right) =$$

$$= \left(G(k, t_0)\widetilde{\Phi}\left(\frac{t_0}{q^k}, t_0\right) + \sum_{j=1}^{k-1} A_j\widetilde{\Phi}\left(\frac{t_0}{q^{k-j}}, t_0\right)\right)\widetilde{x}(t_0) +$$

$$+ \left([A_k, \ldots, A_{k+l}]\right)\widetilde{x}(t_0) = \widetilde{\Phi}(k+1, t_0)\widetilde{x}(t_0).$$

Hence, from the principle of mathematical induction the formula (8) holds for all $k \in \mathbb{N} \cup \{0\}$.

Remark 2.8. If $l = 0$ then the memory $M(0, t_0, \varphi)$ is only one-valued, i.e. $M(0, t_0, \varphi) = \varphi(t_0)$. Moreover, in that case, matrix $\widetilde{\Phi}(k, t_0) = \Phi(k, t_0)$ and

$$\Phi(k+1, t_0) = G(k, t_0)\Phi(k, t_0) + \sum_{j=1}^{k} A_j\Phi(k-j, t_0).$$

Additionally $\gamma\left(\frac{t_0}{q^k}, t_0, M(0, t_0, \varphi)\right) = \Phi(k, t_0)\varphi(t_0)$.

3. Observability in finite memory domain

The standard definition of observability says that a system is observable on a time-interval if from the knowledge of the output one can reconstruct uniquely the initial condition. As we consider systems together with the l-memory, (i.e. the extended initial conditions), we want to determine the extended initial condition $\widetilde{x}(t_0)$ from the knowledge of the sequence of outputs

$$\mathcal{Y} := \left\{y\left(\frac{t_0}{q^k}\right), k = 0, \ldots s\right\}.$$

Hence, the starting point t_0 is important (similarly like it is for time-varying systems). For that reason, following by [17, 18], a pair $(t, \widetilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^{n+nl}$ is called an l-event.

Let us consider the linear autonomous q-difference fractional-order system $\Sigma_{(\alpha,q,l)}$ given by equations (3)–(5).

Definition 3.1. Let $l, s \in \mathbb{N} \cup \{0\}$ and $t_0 > 0$. Let $\varphi_{1,2} : \mathbb{R} \to \mathbb{R}^n$. We say that two l-events $(t_0, \widetilde{x}_1)$, $(t_0, \widetilde{x}_2)$, where $\widetilde{x}_1 = M(l, t_0, \varphi_1)$, $\widetilde{x}_2 = M(l, t_0, \varphi_2)$, are indistinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps if and only if

$$C\gamma(t, t_0, \widetilde{x}_1) = C\gamma(t, t_0, \widetilde{x}_2) \tag{9}$$

for all $t \in \Omega_s\left(\frac{t_0}{q^s}\right)$. Otherwise, the l-events $(t_0, \widetilde{x}_1)$, $(t_0, \widetilde{x}_2)$ are distinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps.

The next proposition follows directly from the above definition.

Proposition 3.2. Two l-events $(t_0, \widetilde{x}_1)$, $(t_0, \widetilde{x}_2)$ are indistinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps if and only if

$$C\widetilde{\Phi}(k, t_0)\widetilde{x}_1 = C\widetilde{\Phi}(k, t_0)\widetilde{x}_2 \tag{10}$$

for all $k \in \{0, \ldots, s\}$.

As (10) can be written in the form

$$C\widetilde{\Phi}(k, t_0)\left(\widetilde{x}_1 - \widetilde{x}_2\right) = 0,$$

then the following statements are equivalent:

a) The events $(t_0, \widetilde{x}_1), (t_0, \widetilde{x}_2) \in \mathbb{R}_+ \times \mathbb{R}^{n+ln}$ are indistinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps.
b) The events $(t_0, \widetilde{x}_1 - \widetilde{x}_2), (t_0, \mathbf{0}) \in \mathbb{R}_+ \times \mathbb{R}^{n+ln}$ are indistinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps.

Definition 3.3. Let $l, s \in \mathbb{N} \cup \{0\}$ and $t_0 > 0$, $\varphi_{1,2} : \mathbb{R} \to \mathbb{R}^n$. We say that $\Sigma_{(\alpha,q,l)}$ is observable at t_0 in s-steps if any two l-events $(t_0, \widetilde{x}_1)$, $(t_0, \widetilde{x}_2)$, where $\widetilde{x}_1 = M(l, t_0, \varphi_1)$, $\widetilde{x}_2 = M(l, t_0, \varphi_2)$, are distinguishable with respect to the system $\Sigma_{(\alpha,q,l)}$ in s-steps.

Proposition 3.4. The system $\Sigma_{(\alpha,q,l)}$ is observable at t_0 in s-steps if and only if the initial extended state $\widetilde{x}(t_0) = M(l, t_0, \varphi)$ can be uniquely determined from the knowledge of the sequence of outputs $\mathcal{Y} = \left\{y\left(\frac{t_0}{q^k}\right), k = 0, \ldots, s\right\}$.

Proof. "$\Rightarrow$" Let us assume that there are two l-memories such that for all $k \in \{0, \ldots, s\}$ we have:

$$C\gamma\left(\frac{t_0}{q^k}, t_0, \widetilde{x}_1\right) = C\gamma\left(\frac{t_0}{q^k}, t_0, \widetilde{x}_2\right) = y\left(\frac{t_0}{q^k}\right).$$

It means that $\widetilde{x}_1$, $\widetilde{x}_2$ are indistinguishable with respect to $\Sigma_{(\alpha,q,l)}$ in s-steps. This contradicts to observability.

"$\Leftarrow$" Let us assume again that there are two l-memories such that $C\gamma\left(\frac{t_0}{q^k}, t_0, \widetilde{x}_1\right) = C\gamma\left(\frac{t_0}{q^k}, t_0, \widetilde{x}_2\right)$ for all $k \in \{0, \ldots, s\}$. But it is possible only if $\widetilde{x}_1 = \widetilde{x}_2$.

Let us denote by $\mathcal{O}(s)$ the matrix:

$$\mathcal{O}(s) := \begin{bmatrix} C\widetilde{\Phi}(0, t_0) \\ C\widetilde{\Phi}(1, t_0) \\ \vdots \\ C\widetilde{\Phi}(s, t_0) \end{bmatrix}$$

and call it the *observability matrix in s-steps for* $\Sigma_{(\alpha,q,l)}$.

The following Proposition can be proved in the same manner as in the classical linear control theory, for example see [19].

Proposition 3.5. Let $l, s \in \mathbb{N} \cup \{0\}$, and $t_0 > 0$. The system $\Sigma_{(\alpha,q,l)}$ is observable at t_0 in s-steps if and only if one of the following conditions hold:

(i) the $(nl + n) \times (nl + n)$ real matrix:

$$W(s, t_0) = \sum_{k=0}^{s} \widetilde{\Phi}^T(k, t_0) C^T C \widetilde{\Phi}(k, t_0)$$

is nonsingular;

(ii) the matrix $C\widetilde{\Phi}(k, t_0)$ has linearly independent columns for all $k \in \{0, \ldots, s\}$;

(iii) $\operatorname{rank} \mathcal{O}(s) = \operatorname{rank} \begin{bmatrix} C\widetilde{\Phi}(0, t_0) \\ C\widetilde{\Phi}(1, t_0) \\ \vdots \\ C\widetilde{\Phi}(s, t_0) \end{bmatrix} = nl + n.$

Proposition 3.6. If rank $C = n$ and rank

$$\begin{bmatrix} b_2 & \ldots & b_{l+1} \\ \vdots & & \vdots \\ b_s & \ldots & b_{l+s+1} \end{bmatrix} = n,$$

then the system $\Sigma_{(\alpha,q,l)}$ is observable at any t_0 in s-steps.

Proof. Following the construction of matrices $\widetilde{\Phi}(k, t_0)$ we can write that

$$\operatorname{rank} \mathcal{O}(s) = \operatorname{rank} \begin{bmatrix} C\Phi(0, t_0) & \mathbf{0} & \ldots & \mathbf{0} \\ C\Phi(1, t_0) & CA_1 & \ldots & CA_l \\ \vdots & \vdots & \ldots & \vdots \\ C\Phi(s, t_0) & CA_{s+1} & \ldots & CA_{s+l} \end{bmatrix}$$

$$= \operatorname{rank} C + \operatorname{rank} \begin{bmatrix} CA_1 & \ldots & CA_l \\ \vdots & \ldots & \vdots \\ CA_{s+1} & \ldots & CA_{s+l} \end{bmatrix}.$$

Let

$$C \in \mathbb{R}^{p \times n},$$

$$\widetilde{C} = \begin{bmatrix} C & 0 & \ldots & 0 \\ 0 & C & \ldots & 0 \\ 0 & 0 & \ldots & C \end{bmatrix} \in \mathbb{R}^{[(s+1)p] \times [(s+1)n]}$$

and

$$B = \begin{bmatrix} -b_2 I_n & -b_3 I_n & \ldots & -b_{l+1} I_n \\ \vdots & \vdots & \ldots & \vdots \\ -b_{s+2} I_n & -b_{s+3} I_n & \ldots & -b_{s+l+1} I_n \end{bmatrix} \in \mathbb{R}^{[(ls+1)n] \times (nl)}.$$

Notice that rank$C = \min(p, n)$ if and only if rank$\widetilde{C} = (s+1)\min(p, n)$. Then

$$\operatorname{rank} \mathcal{O}(s) = \operatorname{rank} C + \operatorname{rank} \widetilde{C} B.$$

Moreover, if $\operatorname{rank} C = n$, then $\operatorname{rank} \widetilde{C} = (s+1)n$ and $\operatorname{rank} \widetilde{C} B = \operatorname{rank} B$. It means that

$$\operatorname{rank} B = l \operatorname{rank} \begin{bmatrix} b_2 & \ldots & b_{l+1} \\ \vdots & & \vdots \\ b_s & \ldots & b_{l+s+1} \end{bmatrix} = ln.$$

Hence, rank $\mathcal{O}(s) = n + ln$.

Remark 3.7. Let $l = 0$. Then the rank condition of observability matrix takes classical form with $\operatorname{rank} \mathcal{O}(s) = n$.

Example 3.8. Let us consider the system $\Sigma_{(\alpha,q,l)}$ with $n = 1$, i.e.:

$$\Delta_q^\alpha x(t) = ax(t), y(t) = cx(t),$$

where $a, c \in \mathbb{R}$. Then

$$c\widetilde{\Phi}(0, t_0) = [c, 0], \quad \widetilde{\Phi}(1, t_0) = [cG(0, t_0), -b_2 c].$$

Moreover, for $c \neq 0$ we have that $\operatorname{rank} \mathcal{O}(1) = 2$ and the system is observable in 1-memory in s-steps for any $s \in \mathbb{N}$.

Example 3.9. Let us consider the system $\Sigma_{(\alpha,q,l)}$ with the matrix $A = 0$, so $\Delta_q^\alpha x(t) = 0$ and with the output $y(t) = Cx(t)$. Then for each $k \in \mathbb{N} \cup \{0\}$: $G(k, t_0) = -b_1 I_n$ and the solution of l-memory initial value problem with the set of conditions $x(t) = (\varphi u_a)(t)$, for $a = q^l t_0$ and $t_0 > 0$, in general is not zero. We can also notice that

(i) rank $C = n \Leftrightarrow$ rank$\mathcal{O}(s) = nl + n \Leftrightarrow \Sigma_{(\alpha,q,l)}$ is observable for any $s \geq 1$;

(ii) If rank $C < n$ then for all s system $\Sigma_{(\alpha,q,l)}$ is not observable in s-steps.

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