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Robust stability of positive discrete-time linear systems of fractional order

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Abstract. The paper is devoted to the problem of robust stability of linear positive discrete-time systems of fractional order with structured perturbations of state matrices. Simple necessary and sufficient conditions for robust stability in the general case and in the case of linear uncertainty structure with unity rank uncertainty structure and with non-negative perturbation matrices, are established. It is shown that robust stability of the positive discrete-time fractional system is equivalent to: 1) robust stability of the corresponding positive discrete-time system of natural order - in the general case, 2) robust stability of the corresponding finite family of positive discrete-time systems of natural order - in the case of linear unity rank uncertainty structure, 3) asymptotic stability of only one corresponding positive discrete-time system of natural order – in the case of linear uncertainty structure with non-negative perturbation matrices. Moreover, simple necessary and sufficient condition for robust stability of the positive interval discrete-time linear systems of fractional order is given. The considerations are illustrated by numerical examples.

Key words: stability, robust stability, linear system, fractional, positive, discrete-time, linear uncertainty, interval system.

1. Introduction

A dynamical system is called positive if any trajectory of the system starting from non-negative initial states remains forever non-negative for non-negative controls. An overview of state of the art in positive systems theory is given in the monographs [1, 2].

A dynamical system represented by differential (or difference) equations with not necessarily integer orders of derivatives (or differences) can be considered as a fractional order system. The real objects are generally fractional, however, for many of them the fractionality is very low. Therefore, the fractional order representation is more adequate to describe real world systems than the integer order models.

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations has been considered in many papers and monographs, see [3–7], for example.

The new class of linear fractional order systems, namely the positive systems of fractional order has been considered in [8, 9].

The problems of stability and robust stability of the standard fractional order linear systems have been investigated in [10–23].

The problem of stability of positive fractional discrete-time linear systems is addressed in the papers [24–27].

The main purpose of the paper is to give the necessary and sufficient conditions for robust stability of linear positive discrete-time systems of fractional order in the general case and with linear uncertainty structure and in two special cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices.

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To the best knowledge of the author, the robust stability problem of positive fractional discrete-time linear systems has not been considered yet.

In the paper the following notations are used: $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n = \Re_+^{n\times 1}$; $[X^-,\ X^+]$ – the interval matrix; Z_+ – the set of non-negative integers; I_n – the $n\times n$ identity matrix; a vector $x\in\Re^n$ is called strictly positive (strictly negative) and denoted by $x>0$ ($x<0$) if all entries are positive (negative).

2. Problem formulation

Let us consider an uncertain discrete-time linear system of fractional order $\alpha\in(0,\ 1)$, described by the state equation

$$\Delta^\alpha x_{i+1} = A(q)x_i + Bu_i, \quad 0<\alpha<1, \quad i\in Z_+, \tag{1}$$

where $x_i\in\Re^n$ is the state vector, $u_i\in\Re^{n_u}$ is the input vector, $\Delta^\alpha x_i$ is the fractional difference of order $\alpha\in(0,\ 1)$, defined by [8]

$$\Delta^\alpha x_i = x_i + \sum_{j=1}^{i}(-1)^j \begin{pmatrix}\alpha\\ j\end{pmatrix} x_{i-j} \tag{2}$$

with

$$\begin{pmatrix}\alpha\\ j\end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \qquad j>0, \tag{3}$$

and $B\in\Re^{n\times n_u}$, $A(q)\in\Re^{n\times n}$ for any fixed $q\in Q$, where $q=[q_1,q_2,\ldots,q_m]^{\mathrm{T}}$ is the vector of uncertain physical parameters $q_1,\ q_2,\ldots,q_m$ and

$$Q=\{q_r : q_r\in[q_k^-,\ q_k^+],\ \ r=1,2,\ldots,m\} \tag{4}$$

with $q_r^-\le 0$, $q_r^+\ge 0$ ($r=1,2,\ldots,m$) is the value set of uncertain parameters.

Using definition (2) of fractional difference we may write the equation (1) in the form

$$x_{i+1} = A_\alpha(q)x_i + \sum_{j=1}^{i} c_j(\alpha)x_{i-j} + Bu_i, \quad q \in Q, \tag{5}$$

where

$$A_\alpha(q) = A(q) + I_n\alpha \tag{6}$$

and

$$c_j = c_j(\alpha) = (-1)^j \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}, \quad j = 1, 2, \ldots. \tag{7}$$

By generalization of the positivity condition of fractional discrete-time system without uncertain parameters [8] to the uncertain parameters case one obtains the following definition and lemma.

Definition 1. An uncertain fractional order system (1) is called positive (internally) if for any $q \in Q$ the following condition holds: $x_i \in \Re_+^n \ \forall\, i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_i \in \Re_+^{n_u}$, $i \in Z_+$.

Lemma 1. An uncertain fractional order system (1) is positive if and only if $B \in \Re_+^{n\times n_u}$ and

$$A_\alpha(q) = A(q) + I_n\alpha \in \Re_+^{n\times n}, \quad \forall\, q \in Q. \tag{8}$$

In the paper we assume that all entries of the matrix $A(q)$ (and hence $A_\alpha(q)$) are continuous functions of uncertain parameters, non-linear or linear.

In the case of linear uncertainty structure, all entries of $A(q)$ are linear continuous functions of uncertain parameters. Therefore, we may write

$$A(q) = A_0 + \sum_{r=1}^{m} q_r E_r, \tag{9}$$

where $A_0 \in \Re^{n\times n}$ and $E_r \in \Re^{n\times n}$ $(r = 1, 2, \ldots, m)$ are the nominal and the perturbation matrices, respectively, such that (8) holds.

The system (1) is called the system with linear unity rank uncertainty structure if

$$rank\, E_r = 1, \quad r = 1, 2, \ldots, m. \tag{10}$$

The system (1) has linear uncertainty structure with non-negative perturbation matrices if

$$E_r \in \Re_+^{n\times n}, \quad r = 1, 2, \ldots, m. \tag{11}$$

The coefficients $c_j = c_j(\alpha)$ defined by (7) are positive for $\alpha \in (0,\ 1)$ and they strongly decrease for increasing j. Therefore, can be assumed in practical problems that j is bounded by some natural number h, called the length of practical implementation [24, 26]. In this case Eq. (5) takes the form

$$x_{i+1} = A_\alpha(q)x_i + \sum_{j=1}^{h} c_j(\alpha)x_{i-j} + Bu_i, \quad q \in Q, \tag{12}$$

with the initial conditions $x_{-i} \in \Re_+^n$, $i = 0, 1, \ldots, h$. The equation (12) describes an uncertain positive discrete-time linear system with h delays in state.

The time-delay system (12) is called the practical realization of fractional system (1), or equivalently, of fractional system (5).

By generalization of the stability definitions of fractional discrete-time system without uncertain parameters [24, 26, 27] to the case of uncertain system (1) one obtains the following definitions.

Definition 2. The positive fractional system (1) is called robustly practically stable if the system (12) is robustly stable, i.e. it is asymptotically stable for all $q \in Q$.

Definition 3. The positive fractional system (1) is called robustly stable if the solution x_i of the equation (5) for $B = 0$ satisfies the condition $\lim_{i\to\infty} x_i = 0$ for every non-negative initial conditions and for all $q \in Q$, or equivalently, the system is robustly practically stable for $h \to \infty$.

The practical stability and asymptotic stability problems of positive discrete-time linear systems of fractional order were considered in [24–27].

The aim of this paper is to give simple necessary and sufficient conditions for robust stability of the positive discrete-time fractional system (1) in the general case and in the case of systems with linear uncertainty structure in two sub-cases:

1. unity rank uncertainty structure (the condition (10) holds),
2. non-negative perturbation matrices (the condition (11) holds, satisfaction of (10) is not necessary).

Firstly, we show that robust stability of fractional discrete-time positive system (1) is equivalent to robust stability of the corresponding discrete-time positive system without delays of natural order. Next, we give simple conditions for robust stability.

3. Solution of the problem

Let us consider the positive fractional system

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad 0 < \alpha < 1, \quad i \in Z_+, \tag{13}$$

satisfying the positivity condition

$$A_\alpha = A + I_n\alpha \in \Re_+^{n\times n}. \tag{14}$$

The following theorems and lemma have been proved in [26].

Theorem 1. The positive fractional system (13) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ (i.e. $\lambda > 0$) such that $[D - I_n]\lambda < 0$, where

$$D = A + I_n \in \Re_+^{n\times n}, \tag{15}$$

or equivalently, the positive discrete-time linear system

$$\mathbf{x}_{i+1} = Dx_i, \quad i \in Z_+, \tag{16}$$

is asymptotically stable.

Theorem 2. The positive fractional system (13) is asymptotically stable if and only if one of the following equivalent conditions holds:

1. eigenvalues $z_1, z_2, \cdots, z_n$ of the matrix $D = A + I_n$ have moduli less than 1,
2. all leading principal minors of the matrix $-A$ are positive,

3. all coefficients of the characteristic polynomial of the matrix A are positive.

Lemma 2. The positive fractional system (13) is not asymptotically stable if at least one diagonal entry of the matrix A is positive.

From Theorems 1 and 2 we have the following important remark.

Remark 1. Asymptotic stability of positive fractional system (13) does not depend on the fractional order $\alpha \in (0,\ 1)$.

By generalisation of Theorem 1 to the case of the system (1) with uncertain parameters one obtains the following theorem.

Theorem 3. The positive fractional discrete-time linear system (1) is robustly stable if and only if the positive discrete-time system

$$x_{i+1} = D(q)x_i, \quad q \in Q, \quad i \in Z_+, \tag{17}$$

is robustly stable, where

$$D(q) = A(q) + I_n \in \Re_+^{n\times n},\ \forall\, q \in Q. \tag{18}$$

By generalisation of Theorem 2, Lemma 2 and Remark 1 to the system (1) with uncertain parameters we obtain the following theorem, lemma and remark.

Theorem 4. The positive fractional discrete-time system (1) is robustly stable if and only if the following equivalent conditions hold:

1. all leading principal minors $\Delta_i(q)$ $(i = 1, 2, \ldots, n)$ of the matrix $-A(q)$ are positive for all $q \in Q$, i.e.

$$\min_{q\in Q} \Delta_i(q) > 0, \quad i = 1, 2, \ldots, n, \tag{19}$$

2. all coefficients of the characteristic polynomial of the matrix $A(q)$, of the form

$$w(z, q) = \det(zI_n - A(q)) = z^n + \sum_{i=0}^{n-1} a_i(q)z^i, \tag{20}$$

are positive for all $q \in Q$, i.e.

$$\min_{q\in Q} a_i(q) > 0, \quad i = 0, 1, \ldots, n-1. \tag{21}$$

Lemma 3. The positive fractional discrete-time system (1) is not robustly stable if there exists $q \in Q$ such that at least one diagonal entry of matrix $A(q)$ is positive.

Remark 2. Robust stability of positive fractional system (1) does not depend on the fractional order $\alpha \in (0,\ 1)$, i.e. if this system is positive and asymptotically stable then it is asymptotically stable for all $\alpha \in [\alpha_0,\ 1)$, where

$$\alpha_0 = \max\{-a_{ii}(q) :\ q \in Q,\ i = 1, 2, \ldots, n\} \tag{22}$$

and $a_{ii}(q)$ $(i = 1, 2, \ldots, n)$ are diagonal entries of the matrix $A(q)$. Note that must be $\alpha_0 > 0$, according to the assumption $\alpha \in (0,\ 1)$.

The conditions (19) and (21) can be checked by using the computer programs for minimization with constraints of real multivariable functions.

Example 1. Check robust stability of positive system (1) for $n = m = 2$ with the matrix

$$A(q) = \begin{bmatrix} -0.3 + q_1^2 + q_2 & 0.2 - q_2^2 \\ 0.35 + q_1 - q_2 & -0.3 - q_1 - q_2^2 \end{bmatrix}, \tag{23}$$

where $q \in Q$ and

$$Q = \{q = [q_1,\ q_2]^{\mathrm{T}} : q_r \in [-0.1,\ 0.1],\ \ r = 1,\ 2\}.$$

It is easy to see that the matrix $A_\alpha(q) = A(q) + I_2\alpha$ of the form

$$A_\alpha(q) = \begin{bmatrix} \alpha - 0.3 + q_1^2 + q_2 & 0.2 - q_2^2 \\ 0.35 + q_1 - q_2 & \alpha - 0.3 - q_1 - q_2^2 \end{bmatrix} \tag{24}$$

has all non-negative entries for all $q \in Q$ and all $\alpha \in [\alpha_0,\ 1)$, where from (22) we have $\alpha_0 = 0.39$. Hence, the positivity condition (8) holds for $\alpha \in [0.39,\ 1)$.

Computing the leading principal minors of the matrix $-A(q)$ we obtain:

$$\Delta_1(q) = 0.3 - q_1^2 - q_2, \qquad \Delta_2(q) = \det(-A(q))$$

and

$$\min_{q\in Q} \Delta_1(q) = 0.19 > 0, \qquad \min_{q\in Q} \Delta_2(q) = 0.008 > 0.$$

This means that all leading principal minors of the matrix $-A(q)$ are positive for all $q \in Q$ and the system of fractional order $\alpha \in [0.39,\ 1)$ is robustly stable, according to condition 1) of Theorem 2 and Remark 2.

Now we consider the positive discrete-time system (1) with linear uncertainty structure, i.e. with the state matrix of the form (9). To robust stability analysis of such systems we can use Theorem 4.

It is easy to see that asymptotic stability of the positive fractional nominal system

$$\Delta^\alpha x_{i+1} = A_0 x_i, \quad i \in Z_+, \tag{25}$$

is necessary for robust stability of the positive fractional system (1) with linear uncertainty structure.

To stability analysis of the system (25) we can apply Theorem 2 for $A = A_0$.

From the above and Lemma 2 we have the following lemma.

Lemma 4. The fractional positive system (1) with linear uncertainty structure is not robustly stable if at least one diagonal entry of the nominal matrix A_0 is positive.

Now we consider the system (1) with linear unity rank uncertainty structure (the condition (10) holds). In this case the matrix $A(q)$ (9) has linear unity rank uncertainty structure and all coefficients of the polynomial (20) are real multilinear functions of uncertain parameters.

Let us denote by $\bar{q}_1,\ \bar{q}_2,\ \ldots,\ \bar{q}_K$ $(K = 2^m)$, where $\bar{q}_k = [\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_m]^T$ with $\hat{q}_r = q_r^-$ or $\hat{q}_r = q_r^+$, $r = 1, 2, \ldots, m$, the vertices of hiperrectangle (4).

Moreover, by $V_k = A(\bar{q}_k)$, $\quad k = 1, 2, \ldots, K$, denote the vertex matrices of the family of matrices $\{A(q) :\ \ q \in Q\}$, where $A(q)$ has the form (9). These matrices correspond to the vertices of the set (4).

Theorem 5. The positive fractional discrete-time system (1) with linear unity rank uncertainty structure is robustly stable if and only if all the positive vertex systems

$$\Delta^\alpha x_{i+1} = V_k x_i, \quad k = 1, 2, \ldots, K, \tag{26}$$

are asymptotically stable, i.e. the conditions of Theorem 2 are satisfied for $A = V_k$ and for all $k = 1, 2, \ldots, K$.

Proof. Necessity is obvious because the systems (26) belong to the family (1) of positive systems with $A(q)$ of the form (9).

The proof of sufficiency is based on the following observation: if the system (1) has linear unity rank uncertainty structure (the condition (10) holds) then the coefficients $a_i(q)$, $i = 0, 1, \ldots, n-1$, of (20) are real multilinear functions of uncertain parameters q_r, $r = 1, 2, \ldots, m$, and therefore

$$\min_{q \in Q} a_i(q) = \min_k a_i(\bar{q}_k), \quad i = 0, 1, \ldots, n-1. \tag{27}$$

From the condition 3. of Theorem 2 it follows that if the positive systems (26) are asymptotically stable, then all coefficients of the characteristic polynomials of the matrices V_k, $k = 1, 2, \ldots, K$, are positive, i.e.

$$a_i(\bar{q}_k) > 0, \quad i = 0, 1, \ldots, n-1, \quad k = 1, 2, \ldots, K. \tag{28}$$

Hence, $\min_k a_i(\bar{q}_k) > 0$ for $i = 0, 1, \ldots, n-1$, and by (27),

$$\min_{q \in Q} a_i(q) > 0, \quad i = 0, 1, \ldots, n-1. \tag{29}$$

This means that all coefficients of the polynomial (20) are positive for all $q \in Q$, and by condition 2. of Theorem 4, the positive system (1) is robustly stable.

To asymptotic stability analysis of the positive systems (26) we can apply Theorem 2 assuming $V_k = A(\bar{q}_k)$ for $k = 1, 2, \ldots, K$ instead of the matrix A.

Example 2. Check robust stability of the positive system (1) with $n = 2$, $\alpha = 0.5$, $m = 2$ and the matrix $A(q)$ of the form (9) with

$$A_0 = \begin{bmatrix} -0.3 & 0.15 \\ 0.3 & -0.4 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -0.5 & 0 \end{bmatrix}, \tag{30}$$

where $q \in Q$ with

$$Q = \{q = [q_1, \ q_2]^{\mathrm{T}} : q_r \in [-0.1, \ 0.1], \ \ r = 1, \ 2\}. \tag{31}$$

It is easy to see that the system (1) with the state matrix (9), (30) is positive (the condition (8) holds), has unity rank uncertainty structure (the condition (10) holds) and the positive nominal system (25) is asymptotically stable (all leading principal minors of the matrix $-A_0$ are positive).

We apply Theorem 5 to the robust stability analysis.

The set (31) of $m = 2$ uncertain parameters has $K = 2^m = 4$ vertices. Hence, there is $K = 4$ the vertex systems (26). Asymptotic stability of the vertex systems is necessary and sufficient for robust stability of the system under consideration.

Computing the vertices of the value set (31) of uncertain parameters, the vertex matrices $V_k = A(\bar{q}_k)$ and the matrices $-V_k$, $k = 1, 2, \ldots, 4$, one obtains

$$\bar{q}_1 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}, \quad \bar{q}_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad \bar{q}_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad \bar{q}_4 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \tag{32}$$

$$-V_1 = \begin{bmatrix} 0.5 & -0.15 \\ -0.35 & 0.4 \end{bmatrix}, \quad -V_2 = \begin{bmatrix} 0.3 & -0.15 \\ -0.25 & 0.4 \end{bmatrix}, \tag{33a}$$

$$-V_3 = \begin{bmatrix} 0.1 & -0.15 \\ -0.25 & 0.4 \end{bmatrix}, \quad -V_4 = \begin{bmatrix} 0.3 & -0.15 \\ -0.35 & 0.4 \end{bmatrix}. \tag{33b}$$

It is easy to check that all leading principal minors of matrices (33) are positive. This means, according to condition 2) of Theorem 2, that all the vertex positive systems (26) are asymptotically stable. Hence, from Theorem 5 it follows that the system is robustly stable.

The same result we obtain from Theorem 4, because all leading principal minors of the matrix $-A(q)$ of the form

$$-A(q) = \begin{bmatrix} 0.3 - q_1 - q_2 & -0.15 \\ -0.3 + 0.5q_2 & 0.4 \end{bmatrix},$$

are positive for all $q \in Q$.

Note that the system is positive and robustly stable not only for $\alpha = 0.5$, but for any $\alpha \in [\alpha_0, \ 1)$, where according to (22), $\alpha_0 = \max_{q \in Q}(0.3 - q_1 - q_2) = 0.5$.

Now we consider the special case of the positive discrete-time fractional system (1) with linear uncertainty structure, called the positive fractional discrete-time interval system. This system is described by the following autonomous state equation

$$\Delta^\alpha x_{i+1} = [A^-, \ A^+] x_i, \quad i \in Z_+, \tag{34}$$

where $\alpha \in (0, \ 1)$, $[A^-, \ A^+]$ is the interval matrix, i.e. the set of real $n \times n$ matrices $A = [a_{ij}]$, such that $a_{ij}^- \le a_{ij} \le a_{ij}^+$, $i, j = 1, 2, \ldots, n$, where $A^- = [a_{ij}^-]$, $A^+ = [a_{ij}^+]$.

From the above and Lemma 1 it follows that the system (34) is positive if and only if

$$A_\alpha^- = A^- + I_n \alpha \in \Re_+^{n \times n}. \tag{35}$$

Applying Theorem 3 to the positive system (35) we obtain that robust stability of this system is equivalent to robust stability of the positive interval system of natural order

$$x_{i+1} = [D^-,\ D^+]x_i, \quad i \in Z_+, \tag{36}$$

where

$$\begin{aligned} D^- &= (A^- + I_n) \in \Re_+^{n\times n}, \\ D^+ &= (A^+ + I_n) \in \Re_+^{n\times n}. \end{aligned} \tag{37}$$

It is easy to see that the interval matrix $[D^-,\ D^+]$ is a matrix with non-negative entries if and only if the condition (35) holds.

Theorem 6. The positive fractional discrete-time interval system (34) is robustly stable if and only if the positive system

$$x_{i+1} = D^+ x_i, \quad i \in Z_+, \tag{38}$$

is asymptotically stable.

Proof. It follows from the above considerations and the fact that the positive interval system (36) is robustly stable if and only if the positive system (38) is asymptotically stable [28].

To asymptotic stability checking of the positive systems (38) we can apply Theorem 2 for $A = D^+$.

In the case of the positive system (1) with linear uncertainty structure and non-negative perturbation matrices, i.e. with the state matrix of the form (9) satisfying the condition (11), we have $q_r E_r \in [q_r^- E_r,\ q_r^+ E_r]$ for any fixed $q_r \in [q_r^-,\ q_r^+]$ and $A(q) \in [A^-,\ A^+]$ for all $q \in Q$, where

$$A^- = A_0 + \sum_{r=1}^{m} q_r^- E_r, \quad A^+ = A_0 + \sum_{r=1}^{m} q_r^+ E_r. \tag{39}$$

This means that robust stability of the positive interval system (34) is sufficient for robust stability of the positive system (1) with linear uncertainty structure and non-negative perturbation matrices. Moreover, asymptotic stability of the positive system (38) is necessary and sufficient for robust stability of the positive interval system (34).

Hence, asymptotic stability of the positive system (38) is sufficient for robust stability of the positive system (1) with linear uncertainty structure and non-negative perturbation matrices.

From Theorem 1 we have that asymptotic stability of the positive system (38) is equivalent to asymptotic stability of the positive fractional system (1) with the state matrix $A(q) = A^+$, where A^+ has the form given in (39).

This means that asymptotic stability of the positive system (38) is also necessary for robust stability of the positive system (1) with linear uncertainty structure and non-negative perturbation matrices.

Hence, we have the following theorem and lemma.

Theorem 7. The positive fractional discrete-time system (1) with linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the positive system of natural order (38) is asymptotically stable, where D^+ has the form given in (37).

Lemma 5. The positive system (1) with linear uncertainty structure and non-negative perturbation matrices is not robustly stable if at least one diagonal entry of the matrix A^+ is positive.

From Theorems 2 and 7 we obtain the following theorem.

Theorem 8. The positive fractional discrete-time system (1) with linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if one of the following equivalent conditions holds:

1. eigenvalues $z_1, z_2, \cdots, z_n$ of the matrix $D^+ = A^+ + I_n$ have moduli less than 1,
2. all leading principal minors of the matrix $-A^+$ are positive,
3. all coefficients of the characteristic polynomial of the matrix A^+ are positive.

Example 3. Check robust stability of the positive fractional system (1) with $n = 2$, $\alpha = 0.5$, $m = 2$ and the matrix $A(q)$ of the form (9) with

$$A_0 = \begin{bmatrix} -0.3 & 0.15 \\ 0.3 & -0.4 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix}, \tag{40}$$

where the set Q is given by (31).

The system under consideration is a positive system with linear uncertainty structure with non-negative perturbation matrices. Therefore, we apply Theorem 8 to the robust stability analysis.

Computing the matrix $-A^+$, where A^+ has the form given in (39) we obtain

$$-A^+ = \begin{bmatrix} 0.1 & -0.15 \\ -0.35 & 0.4 \end{bmatrix}. \tag{41}$$

Matrix (41) has non-positive leading principal minor $\Delta_2 = \det(-A^+)$ and the system is not robustly stable, according to Theorem 8.

The same result we obtain from Theorem 4, because not all leading principal minors of the matrix $-A(q)$ of the form

$$-A(q) = \begin{bmatrix} 0.3 - q_1 - q_2 & -0.15 \\ -0.3 - 0.5q_2 & 0.4 \end{bmatrix},$$

are positive for all $q \in Q$.

4. Concluding remarks

Simple necessary and sufficient conditions for robust stability of the positive discrete-time linear system (1) of fractional order $0 < \alpha < 1$ in the general case and in the case of system with linear uncertainty structure in two sub-cases: 1) unity rank uncertainty structure (the condition (10) holds), 2) non-negative perturbation matrices (the condition (11) holds, satisfaction of (10) is not necessary), have been given.

It has been shown that:

- robust stability of the positive fractional system (1) is equivalent to robust stability of the positive discrete-time system of natural order (17) (Theorem 3),
- the positive fractional system (1) with linear unity rank uncertainty structure is robustly stable if and only if the positive vertex systems (26) of natural order are asymptotically stable (Theorem 5),

- the positive interval fractional system (34) is robustly stable if and only if the positive system (38) of natural order is asymptotically stable (Theorem 6).
- the positive fractional discrete-time system (1) with linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the positive system (38) of natural order is asymptotically stable (Theorem 7).

The proposed conditions for robust stability of positive fractional discrete-time linear systems have been obtained by extension of asymptotic stability conditions given in [26].

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