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Stability and stabilization of positive fractional linear systems by state-feedbacks

T. KACZOREK*

Faculty of Electrical Engineering, Bialystok University of Technology, 45D Wiejska St., 15-351 Białystok, Poland

Abstract. Notations of the practical stability and of the asymptotic stability of positive and cone fractional 1D and 2D linear systems are introduced. Necessary and sufficient conditions for the practical stability and the asymptotic stability of positive and cone fractional 1D and 2D linear systems are established. It is shown that the checking of the practical stability and asymptotic stability of positive 2D linear systems can be reduced to testing the stability of corresponding 1D positive linear systems. Three LMI approaches are proposed for checking the stability of positive fractional linear systems. LMI approach is applied to compute gain matrices of state-feedbacks such that closed-loop systems are positive and asymptotically stable. The proposed methods are illustrated on numerical examples.

Key words: cone fractional, 1D and 2D systems, positive, practical, asymptotical, stability, stabilization, state-feedback, LMI approach.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. The notation of cone systems has been introduced in [3, 4].

The most popular models of two-dimensional (2D) linear systems are the discrete models introduced by Roesser [5], Fornasini-Marchesini [6] and Kurek [7]. The models have been extended for positive systems in [2, 8–10]. Reachability and minimum energy control of standard and positive 2D linear systems have been considered in [8, 11–13].

The notion of internally positive 2D system (model) with delays in states and in inputs has been introduced and necessary and sufficient conditions for the internal positivity, reachability, controllability, observability and the minimum energy control problem have been established in [8, 9, 13].

Stability of positive 1D and 2D linear systems has been considered in [10, 14–18] and the robust stability in [19]. Mathematical fundaments of fractional calculus are given in the monographs [20–25]. The positive fractional linear systems have been addressed in [26, 27] and their stability has been investigated in [16, 28–30]. LMI approaches to checking the stability of positive 2D systems have been proposed in [17, 31]. The positive fractional linear 2D systems have been introduced in [32–34]. The concept of practical stability for positive fractional 1-D discrete-time linear systems has been introduced in [29]. Some applications of fractional calculus are given in [24, 35–37].

In this paper the stability and stabilization of positive fractional linear systems by state-feedback will be addressed.

The paper is organized as follows. In Sec. 2 the basic definitions and theorems concerning stability of 1D positive fractional linear systems are recalled and the notation of the practical and asymptotical stability of fractional and cone systems are introduced. The practical and asymptotical stability of 2D positive fractional linear systems are considered in Sec. 3. Necessary and sufficient conditions for the stability are established and it is shown that the checking of the stability of 2D positive linear systems can be reduced to testing the stability of corresponding 1D positive systems. In Sec. 4 the LMI approaches are proposed for testing the stability of the positive fractional linear systems and computation gain matrices of state-feedbacks so that the closed-loop systems are positive and asymptotically stable. Concluding remarks are given in Sec. 5.

In this paper the following notation is used.

The set of real $n \times m$ matrices with nonnegative entries are denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. A matrix $A = [a_{ij}] \in \Re_+^{n\times m}$ (a vector x) is called strictly positive and denoted by $A > 0$ ($x > 0$) if $a_{ij} > 0$ for $i = 1, \dots, n, j = 1, \dots, m$. The set of nonnegative integers will be denoted by Z_+. The $n \times n$ identity matrix is denoted by I_n.

2. Stability of 1D positive fractional linear systems

2.1. Positive 1D systems. Consider the linear discrete-time system:

$$x_{i+1} = Ax_i + Bu_i \tag{1a}$$

$$y_i = Cx_i + Du_i \tag{1b}$$

*e-mail: kaczorek@isep.pw.edu.pl

where, $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$, $i \in Z_+$ are the state, input and output vectors and, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 1. The system (1) is called (internally) positive if $x_i \in \Re^n_+$, $y_i \in \Re^p_+$, $i \in Z_+$ for any $x_0 \in \Re^n_+$ and every $u_i \in \Re^m_+$, $i \in Z_+$.

Theorem 1 [1, 2]. The system (1) is positive if and only if

$$A \in \Re^{n\times n}_+, \quad B \in \Re^{n\times m}_+, \quad C \in \Re^{p\times n}_+, \quad D \in \Re^{p\times m}_+. \tag{2}$$

The positive system (1) is called asymptotically stable if the solution

$$x_i = A^i x_0 \tag{3}$$

of the equation

$$x_{i+1} = Ax_i, \quad A \in \Re^{n\times n}_+, \quad i \in Z_+ \tag{4}$$

satisfies the condition

$$\lim_{i\to\infty} x_i = 0 \quad \text{for every} \quad x_0 \in \Re^n_+. \tag{5}$$

Theorem 2 [1, 16]. For the positive system (4) the following statements are equivalent:

1. The system is asymptotically stable,
2. Eigenvalues $z_1, z_2, \ldots, z_n$ of the matrix A have moduli less 1, i.e. $|z_k| < 1$ for $k = 1, \ldots, n$,
3. $\det[I_n z - A] \neq 0$ for $|z| \geq 1$,
4. $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of the matrix A defined by $\rho(A) = \max_{1\leq k\leq n} \{|z_k|\}$
5. All coefficients $\widehat{a}_i, \quad i = 0, 1, \ldots, n-1$ of the characteristic polynomial
$$p_{\widehat{A}}(z) = \det[I_n z - \widehat{A}] = z^n + \widehat{a}_{n-1}z^{n-1} + \ldots + \widehat{a}_1 z + \widehat{a}_0 \tag{6}$$
of the matrix $\widehat{A} = A - I_n$ are positive,
6. All leading principal minors of the matrix
$$\overline{A} = I_n - A = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n1} & \overline{a}_{n2} & \cdots & \overline{a}_{nn} \end{bmatrix} \tag{7a}$$
are positive, i.e.,
$$|\overline{a}_{11}| > 0, \quad \begin{vmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{vmatrix} > 0, \ \ldots, \det \overline{A} > 0 \tag{7b}$$
7. There exists a strictly positive vector $\overline{x} > 0$ such that
$$[A - I_n]\,\overline{x} < 0. \tag{8}$$

Theorem 3 [2]. The positive system (4) is unstable if at least one diagonal entry of the matrix A is greater than 1.

2.2. Positive fractional systems. The following definition of the fractional difference

$$\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \quad 0 < \alpha < 1 \tag{9}$$

is used, where $\alpha \in R$ is the order of the fractional difference, and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j\,!} & \text{for } j = 1, 2, \ldots \end{cases} \tag{10}$$

Consider the fractional discrete linear system, described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \tag{11a}$$

$$y_k = Cx_k + Du_k, \tag{11b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$, $k \in Z_+$ are the state, input and output vectors and, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$. Using (9) we may write the equations (11) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{12a}$$

$$y_k = Cx_k + Du_k. \tag{12b}$$

Definition 2. The system (12) is called the (internally) positive fractional system if and only if $x_k \in \Re^n_+$ and $y_k \in \Re^p_+$ $\quad k \in Z_+$ for any initial conditions $x_0 \in \Re^n_+$ and all input sequences $u_k \in \Re^m_+$, $k \in Z_+$.

Theorem 4 [26]. The solution of equation (12a) is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} Bu_i, \tag{13}$$

where Φ_k is determined by the equation

$$\Phi_{k+1} = (A + I_n\alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{14}$$

with $\Phi_0 = I_n$.

Lemma 1. [26] If

$$0 < \alpha \leq 1 \tag{15}$$

then

$$(-1)^{i+1} \binom{\alpha}{i} > 0 \quad \text{for} \quad i = 1, 2, \ldots \tag{16}$$

Theorem 5 [26]. Let $0 < \alpha < 1$. Then the fractional system (12) is positive if and only if

$$\begin{gathered} A + I_n\alpha \in \Re^{n\times n}_+, \qquad B \in \Re^{n\times m}_+, \\ C \in \Re^{p\times n}_+, \qquad D \in \Re^{p\times m}_+. \end{gathered} \tag{17}$$

2.3. Practical stability of fractional systems. From (10) and (16) it follows that the coefficients

$$c_j = c_j(\alpha) = (-1)^j \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}, \quad j = 1, 2, \dots \tag{18}$$

strongly decrease for increasing j and they are positive for $0 < \alpha < 1$. In practical problems it is assumed that j is bounded by some natural number h.

In this case the equation (12a) takes the form

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{h} c_j x_{k-j} + B u_k, \quad k \in Z_+, \tag{19}$$

where

$$A_\alpha = A + I_n \alpha. \tag{20}$$

Note that the equations (19) and (12a) describe a linear discrete-time system with h delays in state.

Definition 3. The positive fractional system (12) is called practically stable if and only if the system (19) is asymptotically stable.

Defining the new state vector

$$\widetilde{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix} \tag{21}$$

we may write the equations (19) and (12b) in the form

$$\widetilde{x}_{k+1} = \widetilde{A}\widetilde{x}_k + \widetilde{B} u_k, \quad k \in Z_+, \tag{22a}$$

$$y_k = \widetilde{C} x_k + \widetilde{D} u_k, \tag{22b}$$

where

$$\widetilde{A} = \begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \in \Re_+^{\widetilde{n}\times\widetilde{n}},$$

$$\widetilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re_+^{\widetilde{n}\times m}, \tag{22c}$$

$$\widetilde{C} = \begin{bmatrix} C & 0 & \dots & 0 \end{bmatrix} \in \Re_+^{p\times\widetilde{n}},$$

$$\widetilde{D} = D \in \Re_+^{p\times m}, \qquad \widetilde{n} = (1+h)n.$$

To test the practical stability of the positive fractional system (12) the conditions of Theorem 2 can be used to the system (22).

Theorem 6. The positive fractional system (12) is practically stable if and only if one of the following equivalent conditions is satisfied:

1. Eigenvalues $\widetilde{z}_k$, $k = 1, \dots, \widetilde{n}$ of the matrix $\widetilde{A}$ have moduli less than 1, i.e.

$$|\widetilde{z}_k| < 1 \quad \text{for} \quad k = 1, \dots, \widetilde{n}, \tag{23}$$

2. $\det[I_{\widetilde{n}} z - \widetilde{A}] \neq 0$ for $|z| \geq 1$,
3. $\rho(\widetilde{A}) < 1$where $\rho(\widetilde{A})$ is the spectral radius of the matrix $\widetilde{A}$ defined by $\rho(\widetilde{A}) = \max\limits_{1\leq k\leq\widetilde{n}} \{|\widetilde{z}_k|\}$,
4. All coefficients $\widetilde{a}_i$, $i = 0, 1, \dots, \widetilde{n}-1$ of the characteristic polynomial

$$\begin{aligned} p_{\widetilde{A}}(z) &= \det[I_n(z+1) - \widetilde{A}] = \\ &= z^n + \widetilde{a}_{n-1} z^{n-1} + \dots + \widetilde{a}_1 z + \widetilde{a}_0 \end{aligned} \tag{24}$$

of the matrix $[\widetilde{A} - I_{\widetilde{n}}]$ are positive,

5. All leading principal minors of the matrix

$$[I_{\widetilde{n}} - A] = \begin{bmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \dots & \widetilde{a}_{1\widetilde{n}} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \dots & \widetilde{a}_{2\widetilde{n}} \\ \dots & \dots & \dots & \dots \\ \widetilde{a}_{\widetilde{n}1} & \widetilde{a}_{\widetilde{n}2} & \dots & \widetilde{a}_{\widetilde{n}\widetilde{n}} \end{bmatrix} \tag{25a}$$

are positive , i.e.

$$|\widetilde{a}_{11}| > 0,$$
$$\begin{vmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} \end{vmatrix} > 0, \dots, \det[I_{\widetilde{n}} - \widetilde{A}] > 0 \tag{25b}$$

6. There exist strictly positive vectors $\overline{x}_i \in \Re_+^n$, $i = 0, 1, \dots, h$ satisfying

$$\overline{x}_0 < \overline{x}_1, \quad \overline{x}_1 < \overline{x}_2, \dots, \overline{x}_{h-1} < \overline{x}_h \tag{26a}$$

such that

$$A_\alpha \overline{x}_0 + c_1 \overline{x}_1 + \dots + c_h \overline{x}_h < \overline{x}_0. \tag{26b}$$

Proof. The first five conditions 1)–5) follow immediately from the corresponding conditions of Theorem 2. Using (8) for the matrix $\widetilde{A}$ we obtain

$$\begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_{h-1} \\ \overline{x}_h \end{bmatrix} < \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_h \end{bmatrix}. \tag{27}$$

From (27) the conditions (26) follow.

Theorem 7. If the positive fractional system (12) is asymptotically stable then the sum of entries of every row of the adjoint matrix Adj $[I_{\widetilde{n}} - \widetilde{A}]$ is strictly positive, i.e.

$$\text{Adj}[I_{\widetilde{n}} - \widetilde{A}]^{-1} \mathbf{1}_{\widetilde{n}} > 0, \tag{28}$$

where $\mathbf{1}_{\widetilde{n}} = [\ 1 \quad 1 \quad \dots \quad 1\ ]^T \in \Re_+^{\widetilde{n}}$, T denotes the transpose.

Proof. It is well-known [8, 28] that if the system (22) is asymptotically stable then the vector

$$\overline{x} = [I_{\widetilde{n}} - \widetilde{A}]^{-1} \mathbf{1}_{\widetilde{n}} \tag{29}$$

is its strictly positive ($\overline{x} > 0$) equilibrium point for $\widetilde{B}u = \mathbf{1}_{\widetilde{n}}$. Note that

$$\det\left[I_{\widetilde{n}} - \widetilde{A}\right] > 0 \tag{30}$$

since $\det[I_{\widetilde{n}} - \widetilde{A}] = \widetilde{d}_0$ in (24). The conditions (29) and (30) imply (28).

Example 1. Check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1x_k, \quad k \in Z_+ \tag{31}$$

for $\alpha = 0.5$ and $h = 2$.

Using (18), (20) and (22c) we obtain

$$c_1 = -\frac{\alpha(\alpha-1)}{2} = \frac{1}{8}, \quad c_2 = \frac{1}{16}, \quad A_\alpha = 0.6$$

and

$$\widetilde{A} = \begin{bmatrix} A_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case the characteristic polynomial (24) has the form

$$p_{\widetilde{A}}(z) = \det[I_{\widetilde{n}}(z+1) - \widetilde{A}] = \begin{vmatrix} z+0.4 & -\frac{1}{8} & -\frac{1}{16} \\ -1 & z+1 & 0 \\ 0 & -1 & z+1 \end{vmatrix} = \\ = z^3 + 2.4z^2 + 1.675z + 0.2125. \tag{32}$$

All coefficients of the polynomial (32) are positive and by Theorem 6 the system (31) is practically stable.

Using (28) we obtain

$$\text{Adj}[I_{\widetilde{n}} - \widetilde{A}]\mathbf{1}_{\widetilde{n}} = \\ = \left(\text{Adj}\begin{bmatrix} 0.4 & -\frac{1}{8} & -\frac{1}{16} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2500 \\ 1.4625 \\ 1.6750 \end{bmatrix}$$

and the condition (28) is satisfied.

Theorem 8. The positive fractional system (12) is practically stable only if the positive system

$$x_{k+1} = A_\alpha x_k, \quad k \in Z_+ \tag{33}$$

is asymptotically stable.

Proof. From (26b) we have

$$(A_\alpha - I_n)\overline{x}_0 + c_1\overline{x}_1 + \ldots + c_h\overline{x}_h < 0. \tag{34}$$

Note that the inequality (34) may be satisfied only if there exists a strictly positive vector $\overline{x}_0 \in \Re_+^n$ such that

$$(A_\alpha - I_n)\overline{x}_0 < 0 \tag{35}$$

since $c_1\overline{x}_1 + \ldots + c_h\overline{x}_h > 0$.

By Theorem 2 the condition (35) implies the asymptotic stability of the positive system (33).

From Theorem 8 we have the following important corollary.

Corollary 1. The positive fractional system (12) is unstable for any finite h if the positive system (33) is unstable.

Theorem 9. The positive fractional system (12) is unstable if at least one diagonal entry of the matrix A_α is greater than 1.

Proof. The proof follows immediately from Theorems 8 and 3.

Example 2. Consider the autonomous positive fractional system described by the equation

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.5 & 1 \\ 2 & 0.5 \end{bmatrix} x_k, \quad k \in Z_+ \tag{36}$$

for $\alpha = 0.8$ and any finite h.

In this case $n = 2$ and

$$A_\alpha = A + I_n\alpha = \begin{bmatrix} 0.3 & 1 \\ 2 & 1.3 \end{bmatrix}. \tag{37}$$

By Theorem 9 the positive fractional system is unstable for any finite h since the entry (2,2) of the matrix (37) is greater than 1.

The same result follows from the condition 5 of Theorem 2 since the characteristic polynomial of the matrix $A_\alpha - I_n$

$$p_{\widetilde{A}}(z) = \det[I_{\widetilde{n}}(z+1) - A_\alpha] = \\ = \begin{bmatrix} z+0.7 & -1 \\ -2 & z-0.3 \end{bmatrix} = z^2 + 0.4z - 2.21$$

has one negative coefficient ($\widehat{a}_0 = -2.21$).

2.4. Asymptotic stability of fractional systems.

In this section the practical stability of the positive systems for $h \to \infty$ is addressed.

Definition 4. The positive fractional system (12) is called asymptotically stable if the system is practically stable for $h \to \infty$.

Lemma 2. If $0 < \alpha < 1$ then

$$\sum_{j=1}^{\infty} c_j = 1 - \alpha, \tag{38}$$

where the coefficients c_j are defined by (18).

Proof. Using the Maclaurin series it is easy to show that $(1-z)^\alpha = \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} z^j$ and substituting $z = 1$ we obtain $\sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} = 0$. From this equality and (38) we have

$$1 - \alpha + \sum_{j=2}^{\infty}(-1)^j \binom{\alpha}{j} = \\ = 1 - \alpha - \sum_{j=1}^{\infty}(-1)^j \binom{\alpha}{j+1} = 1 - \alpha - \sum_{j=1}^{\infty} c_j = 0.$$

Theorem 10. The positive fractional system (12) is asymptotically stable if and only if positive system

$$x_{i+1} = (A + I_n)x_i \tag{39}$$

is asymptotically stable.

Proof. It is well-known [14] that the positive system (19) for $h \to \infty$ is asymptotically stable if and only if the positive system

$$x_{i+1} = \left(A_\alpha + \sum_{j=1}^{\infty} c_j I_n \right) x_i \tag{40}$$

is asymptotically stable. The positive systems (39) and (40) are equivalent since by (38) and (20)

$$A_\alpha + \sum_{j=1}^{\infty} c_j I_n = A + I_n \alpha + I_n(1-\alpha) = A + I_n.$$

Applying to the positive system (39) Theorem 6 we obtain the following theorem.

Theorem 11. The positive fractional system (12) is asymptotically stable if and only if one of the equivalent conditions holds:

1. Eigenvalues $z_1, z_2, \ldots, z_k$ of the matrix $A+I_n$ have moduli less than 1, i.e. $|z_k| < 1$ for $k = 1, \ldots, n$,
2. All coefficients of the characteristic polynomial of the matrix A are positive,
3. All leading principal minors of the matrix $-A$ are positive.

Theorem 12. The positive fractional system (12) is unstable if at least one diagonal entry of the matrix A is positive.

Proof. If at least one diagonal entry of the matrix A is positive then at least one diagonal entry of the matrix $A + I_n$ is greater than 1 and it is well-known [2, 16] that the system is unstable.

Example 3. Using Theorem 11 find values of the coefficient c for which the positive fractional system (12) with

$$A = \begin{bmatrix} -0.5 & 1 \\ 0.2 & c \end{bmatrix} \quad \text{and} \quad \alpha = 0.8 \tag{41}$$

is asymptotically stable.

The fractional system is positive if all entries of the matrix

$$A_\alpha = A + I_n\alpha = \begin{bmatrix} 0.3 & 1 \\ 0.2 & c+\alpha \end{bmatrix} \tag{42}$$

are nonnegative, i.e. $c + \alpha \geq 0$ and $c \geq -\alpha = -0.8$.

Applying the condition 2) of Theorem 11 to the matrix (41) we obtain

$$\det[I_n z - A] = \begin{vmatrix} z+0.5 & -1 \\ -0.2 & z-c \end{vmatrix} = $$
$$= z^2 + (0.5-c)z - (0.5c+0.2)$$

and $c < -0.4$. Therefore, the fractional system (12) with (41) is positive and asymptotically stable for $-0.8 \leq c < -0.4$. The same result we obtain using the condition 3) of Theorem 11.

2.5. Cone fractional systems. Definition 5 [3, 4]. Let

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \Re^{n \times n}$$

be nonsingular and p_k be the $k-$th ($k = 1, \ldots, n$) its row. The set

$$\mathcal{P} := \left\{ x \in \Re^n : \bigcap_{k=1}^{n} p_k x \geq 0 \right\} \tag{43}$$

is called a linear cone generated by the matrix P.

In a similar way we may define for the inputs u the linear cone

$$\mathcal{Q} := \left\{ u \in \Re^m : \bigcap_{k=1}^{m} q_k u \geq 0 \right\} \tag{44}$$

generated by the nonsingular matrix

$$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in \Re^{m \times m}$$

and for the outputs y, the linear cone

$$\mathcal{V} := \left\{ y \in \Re^p : \bigcap_{k=1}^{p} v_k y \geq 0 \right\} \tag{45}$$

generated by the nonsingular matrix

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \Re^{p \times p}.$$

Definition 6. The fractional system (12) is called ($\mathcal{P}$, $\mathcal{Q}$, $\mathcal{V}$) cone fractional system if $x_i \in \mathcal{P}$ and $y_i \in \mathcal{V}$, $i \in Z_+$ for every $x_0 \in \mathcal{P}$, $u_i \in \mathcal{Q}$, $i \in Z_+$.

The ($\mathcal{P}$, $\mathcal{Q}$, $\mathcal{V}$) cone fractional system (12) is shortly called the cone fractional system.

Note that if $\mathcal{P} = R_+^n$, $\mathcal{Q} = R_+^m$, $\mathcal{V} = R_+^n$ then the (R_+^n, R_+^m, R_+^p) cone system is equivalent to the classical positive system [3, 4].

Theorem 13. The fractional system (12) is ($\mathcal{P}$, $\mathcal{Q}$, $\mathcal{V}$) cone fractional system if and only if

$$\begin{aligned} \overline{A} &= PAP^{-1} \in \Re_+^{n \times n}, \\ \overline{B} &= PBQ^{-1} \in \Re_+^{n \times m}, \\ \overline{C} &= VCP^{-1} \in \Re_+^{p \times n}, \\ \overline{D} &= VDQ^{-1} \in \Re_+^{p \times m}. \end{aligned} \tag{46}$$

Proof. Let

$$\overline{x}_i = Px_i, \quad \overline{u}_i = Qu_i \quad \text{and} \quad \overline{y}_i = Vy_i, \; i \in Z_+. \tag{47}$$

From definition 5 it follows that if $x_i \in \mathcal{P}$ then $\overline{x}_i \in \Re^n_+$, if $u_i \in \mathcal{Q}$ then $\overline{u}_i \in \Re^m_+$ and if $y_i \in \mathcal{V}$ then $\overline{y}_i \in \Re^p_+$. From (12) and (47) we have

$$\begin{aligned} &\overline{x}_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} \overline{x}_{k-j+1} = \\ &= Px_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} Px_{k-j+1} = \\ &= PAx_k + PBu_k = PAP^{-1}\overline{x}_k + PBQ^{-1}\overline{u}_k = \\ &= \overline{A}\overline{x}_k + \overline{B}\overline{u}_k, \quad k \in Z_+ \end{aligned} \tag{48a}$$

and

$$\begin{aligned} \overline{y}_k &= Vy_k = VCx_k + VDu_k = \\ &= VCP^{-1}\overline{x}_k + VDQ^{-1}\overline{u}_k = \\ &= \overline{C}\overline{x}_k + \overline{D}\overline{u}_k, \quad k \in Z_+. \end{aligned} \tag{48b}$$

It is well-known [2] that the system (48) is the positive one if and only if the conditions (46) are satisfied.

Theorem 14. The cone fractional system (12) is asymptotically stable if and only if the positive fractional system is asymptotically stable.

Proof. From (46) we have

$$\begin{aligned} \det[Iz - \overline{A}] &= \det[Iz - PAP^{-1}] = \det[P(Iz - A)P^{-1}] = \\ &= \det[Iz - A] \det P \det P^{-1} = \det[Iz - A] \end{aligned} \tag{49}$$

since $\det P \det P^{-1} = 1$.

From Theorem 14 we have the following important corollary.

Corollary 2. The cone fractional system (12) is practically stable (asymptotically stable) if and only if the positive fractional system is practically stable (asymptotically stable).

To test the practical stability and the asymptotic stability of the cone fractional system the Theorem 2 and 6 can be used.

3. Stability of 2D positive fractional linear systems

3.1. Positive fractional 2D linear systems. Definition 7 [33]. The (α, β) orders fractional difference of and 2D function x_{ij} is defined by the formula

$$\begin{gathered} \Delta^{\alpha,\beta} x_{ij} = \sum_{k=0}^{i} \sum_{l=0}^{j} c_{\alpha\beta}(k,l) x_{i-k,j-l}, \\ n_1 - 1 < \alpha < n_1, \quad n_2 - 1 < \beta < n_2; \\ n_1, n_2 \in N = \{1, 2, \ldots\}, \end{gathered} \tag{50a}$$

where $\Delta^{\alpha,\beta} x_{ij} = \Delta_i^{\alpha} \Delta_j^{\beta} x_{ij}$ and

$$\begin{aligned} &c_{\alpha,\beta}(k,l) = \\ &= \begin{cases} 1 \quad \text{for } k = 0 \text{ and } l = 0 \\ (-1)^{k+l} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)\beta(\beta-1)\ldots(\beta-l+1)}{k!\,l!} \\ \quad \text{for } k, l \geq 0 \text{ and } k + l > 0 \end{cases}. \end{aligned} \tag{50b}$$

Consider the (α, β) orders fractional 2D linear system, described by the state equations

$$\begin{aligned} \Delta^{\alpha,\beta} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + \\ &+ A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{51a}$$

$$y_{ij} = Cx_{ij} + Du_{ij}, \tag{51b}$$

where $x_{ij} \in \Re^n$, $u_{ij} \in \Re^m$, $y_{ij} \in \Re^p$ are the state, input and output vectors and $A_k \in \Re^{n \times n}$, $B_k \in \Re^{n \times m}$, $k = 0, 1, 2$, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.

Using Definition 7 we may write the equation (51a) in the form

$$\begin{aligned} x_{i+1,j+1} &= \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \\ &- \sum_{k.l \in D_{i+1,j+1} \setminus D_{11}} \sum c_{\alpha,\beta}(k,l) x_{i-k+1,j-l+1} + \\ &+ B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \end{aligned} \tag{52}$$

where $D_{pq} := \{(i,j) : 0 \leq i \leq p, 0 \leq j \leq q,\ i, j \in Z_+\}$ and $\overline{A}_0 = A_0 - I_n \alpha\beta$, $\overline{A}_1 = A_1 + I_n \beta$, $\overline{A}_2 = A_2 + I_n \alpha$.

The boundary conditions for the equation (52) are given by the formula

$$x_{i0},\ i \in Z_+ \quad \text{and} \quad x_{0j},\ j \in Z_+. \tag{53}$$

Definition 8. The system (51) (and also (52)) is called the (internally) positive fractional 2D system if $x_{ij} \in \Re^n_+$ and $y_{ij} \in \Re^p_+$, $i, j \in Z_+$ for any boundary conditions $x_{i0} \in \Re^n_+$, $i \in Z_+$, $x_{0j} \in \Re^n_+$, $j \in Z_+$ and all input sequence $u_{ij} \in \Re^m_+$, $i, j \in Z_+$.

It has been shown in [33] that

a) if $0 < \alpha < 1$ and $1 < \beta < 2$ then

$$\begin{gathered} c_{\alpha,\beta}(k,l) < 0 \quad \text{for} \quad k = 1, 2, \ldots;\ l = 2, 3, \ldots \\ \text{and} \quad c_{\alpha,\beta}(k,1) > 0,\ k = 1, 2, \ldots; \\ c_{\alpha,\beta}(0,l) > 0,\ l = 2, 3, \ldots \end{gathered} \tag{54a}$$

b) if $1 < \alpha < 2$ and $0 < \beta < 1$ then

$$\begin{gathered} c_{\alpha,\beta}(k,l) < 0 \quad \text{for} \quad k = 2, 3, \ldots;\ l = 1, 2, \ldots \\ \text{and} \quad c_{\alpha,\beta}(k,0) > 0,\ k = 2, 3, \ldots; \\ c_{\alpha,\beta}(1,l) > 0,\ l = 1, 2, \ldots. \end{gathered} \tag{54b}$$

Theorem 15 [33]. The fractional 2D linear system (51) for $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) is positive if and only if[1]

$$\begin{gathered} \overline{A}_k \in \Re^{n \times n}_+, \qquad B_k \in \Re^{n \times m}_+, \\ k = 0, 1, 2; \qquad C \in \Re^{p \times n}_+, \qquad D \in \Re^{p \times m}_+. \end{gathered} \tag{55}$$

[1]It is assumed that $\sum_{k=2}^{i+1} c_{k,1} x_{i-k+1,j} = 0$ and $\sum_{l=2}^{j+1} c_{0,l} x_{i+1,j-l+1} = 0$ since $c_{k,1} > 0$, $k = 1, 2, \ldots$ and $c_{0,l} > 0$, $l = 2, 3, \ldots$

3.2. Practical stability. Note that the system (52) is an 2D linear system with the number of delays in state vector increasing to infinity for $i, j \to \infty$.

From (50b) it follows that the coefficients

$$\begin{gathered} c_{k,l} = -c_{\alpha,\beta}(k,l) = \\ = (-1)^{k+l-1} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)\beta(\beta-1)\dots(\beta-l+1)}{k!l!} \\ \text{for} \quad k+l > 0 \end{gathered} \tag{56}$$

strongly decrease for increasing k and l. In practical problems it is assumed that k and l are bounded by some natural numbers L_1 and L_2. In this case the equation (52) for $B_0 = B_1 = B_2 = 0$ takes the form

$$\begin{gathered} x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} + \\ + \sum\sum_{k.l \in D_{i+1,j+1} \setminus D_{11}} c_{k,l} x_{i-k+1,j-l+1} \end{gathered} \tag{57}$$

where $D_{pq} := \{(i,j) : 0 \le i \le p, 0 \le j \le q,\ i,j \in Z_+\}$.

Equation (57) describes an 2D linear system with finite number of delays in state vector. The system (57) has been obtained by neglecting all delays of the system (52) for $i > L_1$ and $j > L_2$.

Define the new state vector

$$\begin{gathered} \widetilde{x}_{ij} = [x_{ij}^T\ x_{i-1j}^T\ \dots\ x_{i-L_1j}^T\ x_{ij-1}^T\ \dots\ x_{i-L_1j-1}^T\ x_{ij-2}^T\ \dots \\ x_{i-L_1j-2}^T\ \dots\ x_{i-L_1j-L_2}^T] \in R^{\widetilde{N}} \\ \widetilde{N} = (L_1+1)(L_2+1)n; \qquad i,j \in Z_+ \end{gathered} \tag{58}$$

we may write Eq. (57) in the form

$$\widetilde{x}_{i+1,j+1} = \widetilde{A}_0 \widetilde{x}_{ij} + \widetilde{A}_1 \widetilde{x}_{i+1,j} + \widetilde{A}_2 \widetilde{x}_{i,j+1} \quad i,j \in Z_+ \tag{59}$$

where

$$\widetilde{A}_0 = \begin{bmatrix} \overline{A}_0 & I_n c_{21} & \dots & I_n c_{L_1,1} & I_n c_{L_1+1,1} & I_n c_{12} & \dots & I_n c_{L_1,2} & I_n c_{L_1+1,2} & I_n c_{13} & \dots & I_n c_{L_1,L_2+1} & I_n c_{L_1+1,L_2+1} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\widetilde{A}_1 = \begin{bmatrix} \overline{A}_1 & 0 & \dots & 0 & 0 & I_n c_{02} & \dots & 0 & 0 & I_n c_{03} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & I_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\widetilde{A}_2 = \begin{bmatrix} \overline{A}_2 & I_n c_{20} & \dots & I_n c_{L_1,0} & I_n c_{L_1+1,0} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix}. \tag{60}$$

Therefore, the fractional 2D system (52) has been reduced to a standard 2D system without delays but with greater dimension.

Theorem 16. The 2D system (59) is positive if and only if

$$\overline{A}_k \in \Re_+^{n\times n}, \qquad k = 0, 1, 2. \tag{61}$$

Proof. Proof follows from (59), (60) and the fact that the system is positive if and only if all matrices have nonnegative entries.

Definition 9. The positive fractional 2D system (51) is called *practically stable* if the system described by the equation (57) is asymptotically stable.

Theorem 17 [16, 18]. The positive fractional 2D system (51) is practically stable if and only if one of the following conditions is satisfied

1.
$$\begin{gathered}\det(I_{\widetilde{N}} - \widetilde{A}_0 z_1 z_2 - \widetilde{A}_1 z_2 - \widetilde{A}_2 z_1) \neq 0 \\ \forall (z_1, z_2) \in B := \{(z_1, z_2) : \, |z_1| \le 1, \; |z_2| \le 1\}.\end{gathered} \tag{62}$$

2. There exists a strictly positive vector $\lambda \in \Re_+^{\widetilde{N}}$ such that
$$[\widetilde{A}_0 + \widetilde{A}_1 + \widetilde{A}_2 - I_{\widetilde{N}}]\lambda < 0. \tag{63}$$

3. The positive 1D system
$$x_{i+1} = (\widetilde{A}_0 + \widetilde{A}_1 + \widetilde{A}_2)x_i, \quad i \in Z_+ \tag{64}$$
is asymptotically stable.

4. The positive 1D system
$$x_{i+1} = \begin{bmatrix} \widetilde{A}_1 + \widetilde{A}_2 & \widetilde{A}_0 \\ I_{\widetilde{N}} & 0 \end{bmatrix} x_i \quad i \in Z_+ \tag{65}$$
is asymptotically stable.

Theorem 18 [16, 18]. The positive fractional 2D system (51) is practically stable only if the positive 2D system

$$\widetilde{x}_{i+1,j+1} = \widetilde{A}_0 \widetilde{x}_{ij} + \widetilde{A}_1 \widetilde{x}_{i+1,j} + \widetilde{A}_2 \widetilde{x}_{i,j+1} \tag{66}$$

is asymptotically stable.

From Theorem 18 we have the following important corollary.

Corollary 3. The positive fractional 2D system (51) is unstable for any finite L_1 and L_2 if the positive 2D system (66) is unstable.

Theorem 19. The positive fractional 2D system (51) is unstable if at least one diagonal entry of the matrix $\overline{A}_1 + \overline{A}_2$ is greater than 1.

Proof. It is well-known [6] that the positive 1D system (65) is asymptotically unstable if at least one diagonal entry of the matrix $\widetilde{A}_1 + \widetilde{A}_2$ is greater than 1. From the structure of the matrices $\widetilde{A}_1$ and $\widetilde{A}_2$ defined by (60) it follows that at least one diagonal entry of the matrix $\widetilde{A}_1 + \widetilde{A}_2$ is greater than 1 if and only if at least one diagonal entry of the matrix $\overline{A}_1 + \overline{A}_2$ is greater than 1. By Theorem 17 the positive fractional 2D system (51) is practically stable if and only if the positive 1D system (70) is asymptotically stable.

Theorem 20. The positive fractional 2D system (51) is unstable if

$$A_k \in \Re_+^{n\times n} \quad \text{for} \quad k = 1, 2. \tag{67}$$

Proof. By Theorem 15 the fractional 2D system (51) for $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) is positive if and only if (55) is satisfied. From (52) it follows that the matrix

$$\overline{A}_1 + \overline{A}_2 = A_1 + A_2 + (\alpha + \beta)I_n \tag{68}$$

has all diagonal entries greater than 1 if (72) holds. In this case by Theorem 19 the positive fractional 2D system (51) is unstable.

3.3. Asymptotic stability.

In this section the practical stability of the positive fractional 2D linear systems for $L_1 \to \infty$ and $L_2 \to \infty$ is addressed.

Definition 10. The positive fractional 2D linear system (51) is called *asymptotically stable* if the system is practically stable for $L_1 \to \infty$ and $L_2 \to \infty$.

In the proof of the main result of this section the following lemma and theorem are used.

Lemma 3. If $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) then

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{\alpha,\beta}(k,l) = 0. \tag{69}$$

Proof. In a similar way as in the proof of Lemma 2 it can be shown that

$$\begin{gathered}\sum_{i=0}^{\infty}(-1)^i \begin{pmatrix} \alpha \\ i \end{pmatrix} = \\ = \sum_{i=0}^{\infty}(-1)^i \frac{\alpha(\alpha-1)\dots(\alpha-i+1)}{i!} = 0 \quad \text{for} \quad \alpha > 0.\end{gathered} \tag{70}$$

Using (50b) and (70) we obtain

$$\begin{gathered}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{\alpha,\beta}(k,l) = \\ = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\alpha(\alpha-1)\dots(\alpha-k+1)\beta(\beta-1)\dots(\beta-l+1)}{k!l!} = \\ \left(\sum_{k=0}^{\infty}(-1)^k\frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}\right)\cdot \\ \cdot\left(\sum_{l=0}^{\infty}(-1)^l\frac{\beta(\beta-1)\dots(\beta-l+1)}{l!}\right) = 0\end{gathered} \tag{71}$$

Theorem 21 [9, 16]. The positive 2D general model with delays

$$\begin{gathered}x_{i+1,j+1} = \sum_{k=0}^{p}\sum_{l=0}^{q}\big(A_{kl}^0 x_{i-k,j-l} \\ + A_{kl}^1 x_{i-k+1,j-l} + A_{kl}^2 x_{i-k,j-l+1}\big) \quad \text{for} \quad i, j \in Z_+,\end{gathered} \tag{72}$$

where $x_{ij} \in \Re_+^n$ is the state vector and $A_{kl}^t \in \Re_+^{n\times n}$, $k = 0,1,\ldots,q$; $l = 0,1,\ldots,p$; $t = 0,1,2$ is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left(\sum_{k=0}^{p}\sum_{l=0}^{q}\left(A_{kl}^0 + A_{kl}^1 + A_{kl}^2\right)\right)x_i \quad \text{for} \quad x_i \in \Re_+^n, \ i \in Z_+ \tag{73}$$

is asymptotically stable.

Theorem 22. The positive fractional 2D system (51) is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left(\widehat{A} + I_n\right)x_i, \qquad \widehat{A} = A_0 + A_1 + A_2, \qquad x_i \in \Re_+^n, \qquad i \in Z_+ \tag{74}$$

is asymptotically stable.

Proof. From (51) for $B_0 = B_1 = B_2 = 0$ we have

$$x_{i+1,j+1} = A_0x_{ij} + A_2x_{i+1,j} + A_2x_{i,j+1} + \sum_{\substack{k=0\\k+l>0}}^{i+1}\sum_{l=0}^{j+1}c_{k,l}x_{i-k+1,j-l+1} \tag{75}$$

where $c_{k,l} = -c_\alpha(k,l)$ for $k + l > 0$.

By Theorem 21 the positive 2D system with delays is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left(\widehat{A} + \sum_{\substack{k=0\\k+l>0}}^{\infty}\sum_{l=0}^{\infty}c_{k,l}I_n\right)x_i, \qquad x_i \in \Re_+^n, \qquad i \in Z_+ \tag{76}$$

is asymptotically stable. From (50b) we have $c_{00} = -1$ and from (74) we obtain

$$\sum_{\substack{k=0\\k+l>0}}^{\infty}\sum_{l=0}^{\infty}c_{k,l}I_n = I_n\,. \tag{77}$$

Substitution of (77) into (76) yields (74).

Applying to the positive 1D system the well-known theorem [16, 28, 29] we obtain the following theorem.

Theorem 23. The positive fractional 2D system (51) is asymptotically stable if and only if one of the following equivalent conditions holds:

1. Eigenvalues $z_1,\ldots,z_n$ of the matrix $\widehat{A} + I_n$ have moduli less than 1.
2. All coefficients of the characteristic polynomial of the matrix $\widehat{A}$ are positive.
3. All leading principles minors of the matrix $-\widehat{A}$ are positive.

Theorem 24. The positive fractional 2D system (51) is unstable if at least one diagonal entry of the matrix $\widehat{A}$ is positive.

Proof. If at least one diagonal entry of the matrix $\widehat{A}$ is positive then at least one diagonal entry of the matrix $\widehat{A} + I_n$ is greater than 1 and it is well-known [3, 6, 9] that the system (74) is unstable.

Example 4. Using Theorem 23 check the asymptotic stability of the positive fractional 2D system (51) for $\alpha = 0.3$ and $\beta = 1.2$ with the matrices

$$A_0 = \begin{bmatrix} 0.4 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0.2 & -1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}. \tag{78}$$

Note that the fractional system is positive since the matrices

$$\overline{A}_0 = A_0 - I_n\alpha\beta = \begin{bmatrix} 0.04 & 0 \\ 0.1 & 0.14 \end{bmatrix}, \quad \overline{A}_1 = A_1 + I_n\beta = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad \overline{A}_2 = A_2 + I_n\alpha = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.4 \end{bmatrix} \tag{79}$$

have nonnegative entries.

In this case

$$\widehat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} -0.8 & 0 \\ 0.5 & -0.5 \end{bmatrix}. \tag{80}$$

The first condition of Theorem 23 is satisfied since the matrix

$$\widehat{A} + I_n = \begin{bmatrix} 0.2 & 0 \\ 0.5 & 0.5 \end{bmatrix} \tag{81}$$

has the eigenvalues $z_1 = 0.2$, $\ z_2 = 0.5$ whose moduli are less than 1.

The second condition of Theorem 23 is also satisfied since characteristic polynomial of the matrix (80)

$$\det[I_nz - \widehat{A}] = \begin{vmatrix} z + 0.8 & 0 \\ -0.5 & z + 0.5 \end{vmatrix} = z^2 + 1.3z + 0.4 \tag{82}$$

has positive coefficients.

All leading principle minors of the matrix

$$-\widehat{A} = \begin{bmatrix} 0.8 & 0 \\ -0.5 & 0.5 \end{bmatrix} \tag{83}$$

are positive, i.e. $\Delta_1 = 0.8$, $\Delta_2 = 0.4$.

Therefore, all three conditions of Theorem 23 are satisfied and the positive fractional 2D system with the matrices (78) is asymptotically stable.

Example 5. Using Theorem 24 we show that the positive fractional 2D system (51) for $\alpha = 0.5$ and $\beta = 1.2$ with the matrices

$$A_0 = \begin{bmatrix} 0.6 & 0.1 \\ 0.1 & 0.7 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -0.1 & 0.3 \\ 0 & -0.2 \end{bmatrix}, \tag{84}$$
$$A_2 = \begin{bmatrix} -0.4 & 0.2 \\ 0 & -0.5 \end{bmatrix}$$

is unstable.

In this case the matrix

$$\widehat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} 0.1 & 0.6 \\ 0.1 & 0 \end{bmatrix} \tag{85}$$

has one positive diagonal entry. Therefore, by Theorem 24 the positive fractional system is unstable. The same result we obtain using one of the conditions of Theorem 23.

4. LMI approaches

4.1. 1D fractional systems. Definition 11 [36]. An inequality of the form

$$\mathrm{F}(x) + F > 0, \tag{86}$$

where x takes values in the real vector space V, the mapping $\mathrm{F} : \mathrm{V} \to S^n$ is linear, and $F \in S^n$, is called the linear matrix inequality (LMI). The LMI is called feasible if there exists an $x \in \mathrm{V}$ such that the inequality (86) is satisfied; otherwise the LMI is called infeasible.

A matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$, $i, j = 1, \ldots, n$. The matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Hurwitz matrix if it has all eigenvalues with negative real parts (the system $\dot{x} = Ax$ is asymptotically stable). The matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Schur matrix if it has all eigenvalues with moduli less than one (the system $x_{i+1} = Ax_i$ is asymptotically stable).

Lemma 4. [17, 31]. A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag\,[P - A^T PA, \quad P] \succ 0 \tag{87}$$

is feasible with respect to the diagonal matrix P.

Lemma 5 [17, 31]. A Metzler matrix $A = \Re^{n\times n}$ is Hurwitz matrix if and only if the LMI

$$blockdiag\,[-(A^T P + PA), \quad P] \succ 0 \tag{88}$$

is feasible with respect to the diagonal matrix P.

It is well-known that $A = \Re_+^{n\times n}$ is Schur matrix if and only if $(A - I_n)$ is Hurwitz matrix.

Lemma 6 [17, 31]. A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag\,\left[-\left((A - I_n)^T P + P(A - I_n)\right), \quad P\right] \succ 0 \tag{89}$$

is feasible with respect to the diagonal matrix P.

Lemma 7. A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag \left\{ \begin{bmatrix} P & -A^T P \\ -PA & P \end{bmatrix}, \; P \right\} \succ 0 \tag{90}$$

is feasible with respect to the diagonal matrix P.

Proof. Consider the congruence transformation

$$\begin{bmatrix} I & A^T \\ 0 & I \end{bmatrix} \begin{bmatrix} P & -A^T P \\ -PA & P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} =$$
$$= \begin{bmatrix} P - A^T PA & 0 \\ 0 & P \end{bmatrix}.$$

It is well-known that the positive definiteness of a matrix is invariant under the congruence transformation. Therefore, the condition (90) is equal to the condition (87).

Theorem 25. The positive fractional system (12) is practically stable if and only if one of the following equivalent conditions holds

1) The LMI

$$blockdiag \left\{ \begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & \ldots & -c_{h-1} A_\alpha^T P_1 & -c_h A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & \ldots & -c_1 c_{h-1} P_1 & -c_1 c_h P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{h-1} P_1 A_\alpha & -c_1 c_{h-1} P_1 & \ldots & P_h - P_{h+1} - c_{h-1}^2 P_1 & -c_{h-1} c_h P_1 \\ -c_h P_1 A_\alpha & -c_1 c_h P_1 & \ldots & -c_{h-1} c_h P_1 & P_{h+1} - c_h^2 P_1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} P_1 & 0 & \ldots & 0 & 0 \\ 0 & P_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & P_h & 0 \\ 0 & 0 & \ldots & 0 & P_{h+1} \end{bmatrix} \right\} \succ 0 \tag{91}$$

is feasible with respect to the diagonal matrices $P_1, \ldots, P_{h+1}$.

2) The LMI

$$blockdiag\left\{-\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & \dots & c_{h-1}P_1 & c_h P_1 \\ P_2 + c_1 P_1 & -2P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{h-1}P_1 & 0 & \dots & -2P_{h-1} & P_{h+1} \\ c_h P_1 & 0 & \dots & P_{h+1} & -2P_h \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \tag{92}$$

is feasible with respect to the diagonal matrices $P_1, \dots, P_{h+1}$.

3) The LMI

$$blockdiag\left\{\begin{bmatrix} P_1 & 0 & \dots & 0 & -A_\alpha^T P_1 & -P_2 & \dots & 0 \\ 0 & P_2 & \dots & 0 & -c_1 P_1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & P_{h+1} & -c_h P_1 & 0 & \dots & -P_{h+1} \\ -P_1 A_\alpha & -c_1 P_1 & \dots & -c_h P_1 & P_1 & 0 & \dots & 0 \\ -P_2 & 0 & \dots & 0 & 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -P_{h+1} & 0 & 0 & \dots & P_{h+1} \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \tag{93}$$

is feasible with respect to the diagonal matrices $P_1, \dots, P_{h+1}$.

The condition 3) can be proved in a similar way using Lemma 7 to the system (22a).

Proof. The positive fractional system (12) is practically stable if and only if the matrix $\overline{A}$ is Schur matrix. Applying to the system (22a) Lemma 4 we obtain the LMI (91), since

$$blockdiag\left[P - \widetilde{A}^T P \widetilde{A}, \quad P\right] = blockdiag\left\{\begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix} - \begin{bmatrix} A_\alpha^T & I_n & \dots & 0 & 0 \\ c_1 I_n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ c_{h-1} I_n & 0 & \dots & 0 & I_n \\ c_h I_n & 0 & \dots & 0 & 0 \end{bmatrix} \cdot\right.$$

$$\left.\cdot \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix} \begin{bmatrix} A_\alpha & c_1 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & I_n & 0 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\}$$

$$= blockdiag\left\{\begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & \dots & -c_{h-1} A_\alpha^T P_1 & -c_h A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & \dots & -c_1 c_{h-1} P_1 & -c_1 c_h P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{h-1} P_1 A_\alpha & -c_1 c_{h-1} P_1 & \dots & P_h - P_{h+1} - c_{h-1}^2 P_1 & -c_{h-1} c_h P_1 \\ -c_h P_1 A_\alpha & -c_1 c_h P_1 & \dots & -c_{h-1} c_h P_1 & P_{h+1} - c_h^2 P_1 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0$$

Similarly, applying to the system (22a) Lemma 6 we obtain the LMI (92), since

$$blockdiag\left[-\left((\overline{A}-I_n)^TP+P(\overline{A}-I_n)\right),\quad P\right]=$$

$$=blockdiag\left\{-\begin{bmatrix} A_\alpha^T-I_n & I_n & \dots & 0 & 0\\ c_1I_n & -I_n & \dots & 0 & 0\\ \vdots & \vdots & \dots & \vdots & \vdots\\ c_{h-1}I_n & 0 & \dots & -I_n & I_n\\ c_hI_n & 0 & \dots & 0 & -I_n\end{bmatrix}\begin{bmatrix} P_1 & 0 & \dots & 0 & 0\\ 0 & P_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & P_h & 0\\ 0 & 0 & \dots & 0 & P_{h+1}\end{bmatrix}\right.$$

$$\left.-\begin{bmatrix} P_1 & 0 & \dots & 0 & 0\\ 0 & P_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & P_h & 0\\ 0 & 0 & \dots & 0 & P_{h+1}\end{bmatrix}\begin{bmatrix} A_\alpha^{-}I_n & c_1I_n & \dots & c_{h-1}I_n & c_hI_n\\ I_n & -I_n & \dots & 0 & 0\\ \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & \dots & -I_n & 0\\ 0 & 0 & \dots & I_n & -I_n\end{bmatrix},\quad \begin{bmatrix} P_1 & 0 & \dots & 0 & 0\\ 0 & P_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & P_h & 0\\ 0 & 0 & \dots & 0 & P_{h+1}\end{bmatrix}\right\}$$

$$=blockdiag\left\{-\begin{bmatrix} A_\alpha^TP_1+P_1A_\alpha-2P_1 & P_2+c_1P_1 & \dots & c_{h-1}P_1 & c_hP_1\\ P_2+c_1P_1 & -2P_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{h-1}P_1 & 0 & \dots & -2P_{h-1} & P_{h+1}\\ c_hP_1 & 0 & \dots & P_{h+1} & -2P_h\end{bmatrix},\quad \begin{bmatrix} P_1 & 0 & \dots & 0 & 0\\ 0 & P_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & P_h & 0\\ 0 & 0 & \dots & 0 & P_{h+1}\end{bmatrix}\right\}\succ 0 \tag{94}$$

Applying to the system (22a) Lemma 7 we obtain the LMI (93) since

$$blockdiag\left\{W,\quad \begin{bmatrix} P_1 & 0 & \dots & 0\\ 0 & P_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & P_{h+1}\end{bmatrix}\right\}\succ 0$$

$$W=\begin{bmatrix} \begin{bmatrix} P_1 & 0 & \dots & 0\\ 0 & P_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & P_{h+1}\end{bmatrix} & \\ -\begin{bmatrix} P_1 & 0 & \dots & 0\\ 0 & P_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & P_{h+1}\end{bmatrix}\begin{bmatrix} A_\alpha & c_1I_n & \dots & c_{h-1}I_n & c_hI_n\\ I_n & 0 & \dots & 0 & 0\\ \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & \dots & I_n & 0\end{bmatrix} & \\ & -\begin{bmatrix} A_\alpha^T & I_n & \dots & 0\\ c_1I_n & 0 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ c_{h-1}I_n & 0 & \dots & I_n\\ c_hI_n & 0 & \dots & 0\end{bmatrix}\begin{bmatrix} P_1 & 0 & \dots & 0\\ 0 & P_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & P_{h+1}\end{bmatrix} \\ & \begin{bmatrix} P_1 & 0 & \dots & 0\\ 0 & P_2 & \dots & 0\\ \vdots & \vdots & \dots & \vdots\\ 0 & 0 & \dots & P_{h+1}\end{bmatrix}\end{bmatrix}.$$

Example 6. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in Z_+ \tag{95}$$

for $\alpha = 0.5$ and $h = 2$.

Using (18) and (22c) we obtain

$$c_1 = \frac{\alpha(1-\alpha)}{2} = \frac{1}{8}, \qquad c_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} = \frac{1}{16},$$

$$A_\alpha = 0.6$$

and

$$\widetilde{A} = \begin{bmatrix} A_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Applying Theorem 25 and MATLAB® environment together with SEDUMI® solver and YALMIP® parser we obtain for the LMI (91)

$$blockdiag \left\{ \begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & -c_2 A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & -c_1 c_2 P_1 \\ -c_2 P_1 A_\alpha & -c_1 c_2 P_1 & P_3 - c_2^2 P_1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \right\} \succ 0$$

where

$$blockdiag\,[P_1, P_2, P_3] =$$

$$= blockdiag\,[\ 7.8921 \quad 3.5026 \quad 2.1132\ ]$$

for LMI (92)

$$blockdiag \left\{ - \begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & c_2 P_1 \\ P_2 + c_1 P_1 & -2P_1 & P_3 \\ c_2 P_1 & P_3 & -2P_2 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \right\} \succ 0$$

where

$$blockdiag\,[P_1, P_2, P_3] =$$

$$= blockdiag\,[\ 6.9266 \quad 3.1155 \quad 2.6096\ ]$$

and for LMI (93)

$$blockdiag \left\{ \begin{bmatrix} P_1 & 0 & 0 & -A_\alpha^T P_1 & -P_2 & 0 \\ 0 & P_2 & 0 & -c_1 P_1 & 0 & -P_3 \\ 0 & 0 & P_3 & -c_2 P_1 & 0 & 0 \\ -P_1 A_\alpha & -c_1 P_1 & -c_2 P_1 & P_1 & 0 & 0 \\ -P_2 & 0 & 0 & 0 & P_2 & 0 \\ 0 & -P_3 & 0 & 0 & 0 & P_3 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \right\} \succ 0$$

where

$$blockdiag\,[P_1, P_2, P_3] =$$

$$= blockdiag\,[\ 7.7203 \quad 3.6738 \quad 2.2765\ ]$$

Therefore, the LMIs are feasible with respect to the matrices P_1, P_2, P_3 and the positive fractional system (95) is practically stable.

Example 7. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.2 & 1 \\ 0.1 & b \end{bmatrix} x_k, \quad k \in Z_+ \tag{96}$$

for $\alpha = 0.8$ and $h = 2$, and the following two values of the coefficient b:

case 1: $b = -0.5$; case 2: $b = 0.5$.

Using (18) and (22c) we obtain

$$c_1 = \frac{\alpha(1-\alpha)}{2} = 0.08,$$

$$c_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} = 0.032$$

and

Case 1. $A_{\alpha_1} = A + I_n\alpha = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 0.3 \end{bmatrix}$

and

$$\widetilde{A}_1 = \begin{bmatrix} A_{\alpha_1} & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 0.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Case 2. $A_{\alpha_2} = A + I_n\alpha = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 1.3 \end{bmatrix}$

and

$$\widetilde{A}_2 = \begin{bmatrix} A_{\alpha_2} & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 1.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In case 1 applying Theorem 25 and MATLAB® environment together with SEDUMI® solver and YALMIP® parser we obtain for the LMI (91)

$$blockdiag\,[P_1, P_2, P_3] =$$
$$= blockdiag\left[\left[\begin{array}{cc} 16.0915 & 0 \\ 0 & 84.3680 \end{array}\right],\right.$$
$$\left.\left[\begin{array}{cc} 4.2540 & 0 \\ 0 & 16.3556 \end{array}\right], \left[\begin{array}{cc} 2.5726 & 0 \\ 0 & 8.6007 \end{array}\right]\right]$$

for LMI (92)

$$blockdiag\,[P_1, P_2, P_3] =$$
$$= blockdiag\left[\left[\begin{array}{cc} 8.8848 & 0 \\ 0 & 35.5971 \end{array}\right],\right.$$
$$\left.\left[\begin{array}{cc} 2.5601 & 0 \\ 0 & 7.2962 \end{array}\right], \left[\begin{array}{cc} 2.2771 & 0 \\ 0 & 5.2364 \end{array}\right]\right]$$

and for LMI (93)

$$blockdiag\,[P_1, P_2, P_3] =$$
$$= blockdiag\left[\left[\begin{array}{cc} 13.3199 & 0 \\ 0 & 70.8279 \end{array}\right],\right.$$
$$\left.\left[\begin{array}{cc} 3.537 & 0 \\ 0 & 13.1042 \end{array}\right], \left[\begin{array}{cc} 2.2117 & 0 \\ 0 & 7.2682 \end{array}\right]\right].$$

In case 2 the positive fractional system (96) is unstable for any h (not only for $h = 2$) since the matrix A_{α_2} has one diagonal entry greater than 1.

The characteristic polynomial of the matrix $A_{\alpha_2} - I_n$

$$p(z) = \det[I_n(z+1) - A_{\alpha_2}] =$$
$$= \left|\begin{array}{cc} z-0.4 & -1 \\ -0.1 & z-0.3 \end{array}\right| = z^2 - 0.7z - 0.22$$

has two negative coefficients. Therefore, the system (96) is also unstable for any h.

4.2. 2D fractional systems. Fractional 2D Roesser model.

The following notions of horizontal and vertical fractional differences of 2D function are used [38].

Definition 12. The α-order horizontal fractional difference of an 2D function x_{ij}, $i, j \in Z_+$ is defined by the formula

$$\Delta_\alpha^h x_{ij} = \sum_{k=0}^{i} c_\alpha(k) x_{i-k,j}, \tag{97a}$$

where $\alpha \in \Re$, $n_1 - 1 < \alpha < n_1 \in N = \{1, 2, \ldots\}$ and

$$c_\alpha(k) = \begin{cases} 1 & \text{for } k = 0 \\ (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} = (-1)^k \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k > 0 \end{cases} \tag{97b}$$

Definition 13. The β-order vertical fractional difference of an 2D function x_{ij}, $i, j \in Z_+$ is defined by the formula

$$\Delta_\beta^v x_{ij} = \sum_{l=0}^{j} c_\beta(l) x_{i,j-l}, \tag{98a}$$

where $\beta \in \Re$, $n_2 - 1 < \beta < n_2 \in N = \{1, 2, \ldots\}$ and

$$c_\beta(l) = \begin{cases} 1 & \text{for } l = 0 \\ (-1)^l \begin{pmatrix} \beta \\ l \end{pmatrix} = (-1)^l \frac{\beta(\beta-1)\ldots(\beta-l+1)}{l!} & \text{for } l > 0 \end{cases}. \tag{98b}$$

Lemma 8 [26]. If $0 < \alpha < 1$ $(0 < \beta < 1)$ then

$$c_\alpha(k) < 0 \quad (c_\beta(k) < 0) \quad \text{for } k = 1, 2, \ldots. \tag{99}$$

Consider the fractional 2D linear system described by the state equations

$$\left[\begin{array}{c} \Delta_\alpha^h x_{i+1,j}^h \\ \Delta_\beta^v x_{i,j+1}^v \end{array}\right] = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} x_{ij}^h \\ x_{ij}^v \end{array}\right] + \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] u_{ij} \tag{100a}$$

$$y_{ij} = \left[\begin{array}{cc} C_1 & C_2 \end{array}\right] \left[\begin{array}{c} x_{i,j}^h \\ x_{i,j}^v \end{array}\right] + D u_{ij} \quad i, j \in Z_+, \tag{100b}$$

where $x_{ij}^h \in \Re^{n_1}$, $x_{ij}^v \in \Re^{n_2}$ are horizontal and vertical state vector at the point (i, j) respectively, $u_{ij} \in \Re^m$ is input vector, $y_{ij} \in \Re^p$ is output vector at the point (i, j) and $A_{11} \in \Re^{n_1 \times n_1}$, $A_{12} \in \Re^{n_1 \times n_2}$, $A_{21} \in \Re^{n_2 \times n_1}$, $A_{22} \in \Re^{n_2 \times n_2}$, $B_1 \in \Re^{n_1 \times m}$, $B_2 \in \Re^{n_2 \times m}$, $C_1 \in \Re^{p \times n_1}$, $C_2 \in \Re^{p \times n_2}$, $D \in \Re^{p \times m}$.

Using Definition 12 and Definition 13 we may write the equation (100a) in the form

$$\begin{gathered} \left[\begin{array}{c} x_{i+1,j}^h \\ x_{i,j+1}^v \end{array}\right] = \left[\begin{array}{cc} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{array}\right] \left[\begin{array}{c} x_{ij}^h \\ x_{ij}^v \end{array}\right] - \\ - \left[\begin{array}{c} \sum\limits_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum\limits_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^h \end{array}\right] + \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] u_{ij}, \end{gathered} \tag{101}$$

where $\overline{A}_{11} = A_{11} + \alpha I_{n_1}$ and $\overline{A}_{22} = A_{22} + \beta I_{n_2}$.

From formula (101) it follows, that the fractional 2D systems are 2D systems with delays increasing with i and j. From (97a) and (98b) it follows that the coefficients $c_\alpha(k)$ and $c_\beta(l)$ in (101) strongly decrease when k and l increase. Therefore, in practical problems we may assume, that k and l are bounded by some natural numbers L_1 and L_2. In this case the equation (101) takes the form

$$\begin{gathered} \left[\begin{array}{c} x_{i+1,j}^h \\ x_{i,j+1}^v \end{array}\right] = \left[\begin{array}{cc} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{array}\right] \left[\begin{array}{c} x_{ij}^h \\ x_{ij}^v \end{array}\right] - \\ - \left[\begin{array}{c} \sum\limits_{k=2}^{L_1+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum\limits_{l=2}^{L_2+1} c_\beta(l) x_{i,j-l+1}^h \end{array}\right] + \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] u_{ij}. \end{gathered} \tag{102}$$

The boundary conditions for the equations (100a), (101) and (102) are given in the form

$$x_{0j}^h \quad \text{for} \quad j \in Z_+, \quad x_{i0}^h \quad \text{for} \quad i \in Z_+ \tag{103}$$

Theorem 26 [38]. The solution of equation (101) with boundary conditions (103) is given by

$$\begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} = \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix} + \\ + \sum_{p=0}^{i}\sum_{q=0}^{j} (T_{i-p-1,j-q}B^{10} + T_{i-p,j-q-1}B^{01})u_{pq}, \tag{104a}$$

where

$$B^{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad B^{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{104b}$$

and the transition matrices $T_{pq} \in \Re^{n\times n}$ are defined by the formula

$$T_{pq} = \begin{cases} I_n & \text{for } p=0, q=0 \\ H & \text{for } p+q>0 \ (p,q \in Z_+) \\ 0 \quad \text{(zero matrix)} & \text{for } p<0 \text{ and / or } q<0 \end{cases} \tag{105a}$$

where

$$H = T_{10}T_{p-1,q} + T_{01}T_{p,q-1} - \sum_{k=2}^{p} [\ c_\alpha(k)I_{n_1} \quad 0\ ]T_{p-k,q} - \\ - \sum_{l=2}^{q} [\ 0 \quad c_\beta(l)I_{n_2}\ ]T_{p,q-l},$$

where

$$T_{10} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & \overline{A}_{22} \end{bmatrix}. \tag{105b}$$

Consider system (102) bounded by two natural numbers L_1 and L_2 and

$$\overline{G}(z_1,z_2) = \begin{bmatrix} I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum\limits_{k=2}^{L_1} c_\alpha(k) z_1^{-k} I_{n_1} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum\limits_{l=2}^{L_2} c_\beta(l) z_2^{-l} I_{n_2} \end{bmatrix}. \tag{106}$$

Let

$$\det \overline{G}(z_1,z_2) = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{N_1-p,N_2-q} z_1^{-p} z_2^{-q}, \tag{107}$$

where N_1, $N_2 \in Z_+$ are determined by the numbers L_1 and L_2 in (102).

Theorem 27 [38]. Let (107) be the characteristic polynomial of the system (102). Then the matrices T_{pq} satisfy the equation

$$\sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{pq}T_{pq} = 0. \tag{108}$$

Theorem 27 is an extension of the well-known classical Cayley-Hamilton theorem for the 2D fractional systems described by the Roesser model (101).

Positive fractional 2D Roesser model

Definition 13. The system (100) is called the (internally) positive fractional 2D system if and only if $x_{ij}^h \in \Re_+^{n_1}$, $x_{ij}^v \in \Re_+^{n_2}$ and $y_{ij} \in \Re_+^p$ $\ i,j \in Z_+$ for any boundary conditions $x_{0j}^h \in \Re_+^{n_1}$, $j \in Z_+$ and $x_{i0}^v \in \Re_+^{n_2}$, $i \in Z_+$ and all inputs $u_{ij} \in \Re_+^p$ $\ i,j \in Z_+$.

Theorem 28 [38]. The fractional 2D system (101) for $\alpha,\beta \in \Re$, $0<\alpha\le 1$, $0<\beta\le 1$ is positive if and only if

$$\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \in \Re_+^{n\times n}, \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n\times m}, \\ [\ C_1 \quad C_2\ ] \in \Re_+^{p\times n}, \qquad D \in \Re_+^{p\times m}. \tag{109}$$

Consider the positive fractional Roesser model (101) with the state-feedback

$$u_{ij} = [\ K_1 \quad K_2\ ] \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} \tag{110}$$

where $K = [\ K_1 \quad K_2\ ] \in \Re^{n\times n}$, $K_j \in \Re^{m\times n_j}$, $j=1,2$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \\ - \begin{bmatrix} \sum\limits_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum\limits_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^h \end{bmatrix} \tag{111}$$

is positive and asymptotically stable.

Theorem 29. The positive fractional closed-loop system (111) is positive and asymptotically stable if and only if there exist a block diagonal matrix

$$\Lambda = blockdiag\,[\Lambda_1,\Lambda_2], \\ \Lambda_k = diag\,[\lambda_{k1},\dots,\lambda_{kn_k}], \\ \lambda_{kj}>0, \quad k=1,2; \quad j=1,\dots,n_k \tag{112}$$

and a real matrix

$$D = [\ D_1 \quad D_2\ ], \quad D_k \in \Re^{m\times n_k}, \quad k=1,2 \tag{113}$$

satisfying conditions

$$\begin{bmatrix} \overline{A}_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & \overline{A}_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \in \Re_+^{n\times n} \tag{114}$$

and

$$\begin{bmatrix} A_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & A_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \begin{bmatrix} 1_{n_1} \\ 1_{n_2} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{115}$$

where $1_{n_k} = [\ 1 \ \dots \ 1\ ]^T \in \Re_+^{n_k}$, k = 1,2.

The gain matrix is given by the formula

$$K = [\ K_1 \quad K_2\ ] = [\ D_1 \quad D_2\ ]\Lambda^{-1} = \\ = [\ D_1\Lambda_1^{-1} \quad D_2\Lambda_2^{-1}\ ]. \tag{116}$$

Proof and a procedure for computation of the gain matrix K are given in [38].

It is well-known [16, 38] that the positive closed-loop system (111) is asymptotically stable if and only if the positive 1D system with the matrix

$$\begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} - \\ -\sum_{k=2}^{\infty}\begin{bmatrix} I_{n_1}c_\alpha(k) & 0 \\ 0 & I_{n_2}c_\beta(k) \end{bmatrix} \tag{117}$$

is asymptotically stable.

Taking into account that [38]

$$\sum_{k=2}^{\infty} c_\alpha(k) = \alpha - 1, \qquad \sum_{k=2}^{\infty} c_\beta(k) = \beta - 1$$

and $\overline{A}_{11} = A_{11} + \alpha I_{n_1}$ and $\overline{A}_{22} = A_{22} + \beta I_{n_2}$ we may write the matrix (117) in the form

$$\begin{bmatrix} \widehat{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \widehat{A}_{22}+B_2K_2 \end{bmatrix} = A + BK \tag{118}$$

where $\widehat{A}_{11} = A_{11} + I_{n_1}$ and $\widehat{A}_{22} = A_{22} + I_{n_2}$ and

$$A = \begin{bmatrix} \widehat{A}_{11} & A_{12} \\ A_{21} & \widehat{A}_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{119}$$

Theorem 30. The fractional closed-loop system (111) is positive and asymptotically stable if and only if there exist a positive definite block diagonal matrix (112) and a real matrix (113) such that the condition (114) is satisfied and the LMI

$$\begin{bmatrix} -\Lambda & A\Lambda + BD \\ (A\Lambda + BD)^T & -\Lambda \end{bmatrix} \prec 0 \tag{120}$$

is feasible with respect to the positive definite diagonal matrix Λ.

Proof. The closed-loop system (111) is positive if and only if the condition (114) is satisfied since the condition

$$\begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} = \\ = \begin{bmatrix} \overline{A}_{11}+B_1D_1\Lambda_1^{-1} & A_{12}+B_1D_2\Lambda_2^{-1} \\ A_{21}+B_2D_1\Lambda_1^{-1} & \overline{A}_{22}+B_2D_2\Lambda_2^{-1} \end{bmatrix} \\ = \begin{bmatrix} \overline{A}_{11}\Lambda_1+B_1D_1 & A_{12}\Lambda_2+B_1D_2 \\ A_{21}\Lambda_1+B_2D_1 & \overline{A}_{22}\Lambda_2+B_2D_2 \end{bmatrix} \cdot \\ \cdot \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_2^{-1} \end{bmatrix} \in \Re_+^{n\times n}$$

is equivalent to (114).

The positive closed-loop system (111) is asymptotically stable if and only if the LMI [17]

$$P - (A+BK)^T P (A+BK) \succ 0 \tag{121}$$

is feasible with respect to a positive definite diagonal matrix P.

Using the Schur complement we can write the condition (121) in the form

$$\begin{bmatrix} -P & P(A+BK) \\ (A+BK)^T P & -P \end{bmatrix} \prec 0. \tag{122}$$

Substitution of (116) and $P = \Lambda^{-1}$ into (122) yields

$$\begin{bmatrix} -\Lambda^{-1} & \Lambda^{-1}(A+BD\Lambda^{-1}) \\ (A+BD\Lambda^{-1})^T\Lambda^{-1} & -\Lambda^{-1} \end{bmatrix} \\ = blockdiag\,[\Lambda^{-1},\Lambda^{-1}] \begin{bmatrix} -\Lambda & A\Lambda+BD \\ (A\Lambda+BD)^T & -\Lambda \end{bmatrix} \\ blockdiag\,[\Lambda^{-1},\Lambda^{-1}] \prec 0. \tag{123}$$

Applying the congruent transformation with the matrix $blockdiag\,[\Lambda,\Lambda]$ we obtain the condition (120).

Example 6. Given the fractional 2D Roesser model with $\alpha = 0.4$, $\beta = 0.5$ and

$$A_{11} = \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & 0.01 \end{bmatrix}, \; A_{12} = \begin{bmatrix} -0.1 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} -0.3 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, \; A_{22} = \begin{bmatrix} -1 & -0.1 \\ 0.4 & 0.1 \end{bmatrix}, \\ B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \; B_2 = \begin{bmatrix} -0.3 \\ 0.2 \end{bmatrix}. \tag{124}$$

Find a gain matrix $K = [\,K_1 \quad K_2\,]$, $K_i \in \Re^{1\times 2}$, $i = 1,2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (124) is not positive since the matrices have negative entries. The model is also unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.01 & 0.2 & 0.1 \\ -0.3 & -0.1 & -1 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.1 \end{bmatrix} \tag{125}$$

has positive diagonal entries.

We choose

$$D = [\,D_1 \quad D_2\,], \qquad D_1 = [\,-0.4 \quad -0.2\,], \\ D_2 = [\,-0.4 \quad -0.2\,]. \tag{126}$$

Applying Theorem 30 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (120) we obtain

$$\Lambda = blockdiag\,[\Lambda_1,\Lambda_2], \\ \Lambda_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.2258 & 0 \\ 0 & 0.2413 \end{bmatrix}. \tag{127}$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (116) we obtain the gain matrix

$$\begin{aligned} K &= [\ K_1 \quad K_2\ ] = [\ D_1\Lambda_1^{-1} \quad D_2\Lambda_2^{-1}\ ] = \\ &= [\ -1 \quad -0.5 \quad -1.7712 \quad -0.8289\ ]. \end{aligned} \tag{128}$$

The closed loop system is positive since matrices

$$\overline{A}_{11} + B_1K_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.36 \end{bmatrix},$$

$$A_{12} + B_1K_2 = \begin{bmatrix} 0.2542 & 0.0658 \\ 0.0229 & 0.0171 \end{bmatrix},$$

$$A_{21} + B_2K_1 = \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix},$$

$$\overline{A}_{22} + B_2K_2 = \begin{bmatrix} 0.0313 & 0.1487 \\ 0.0458 & 0.4342 \end{bmatrix},$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\det \begin{bmatrix} I_{n_1}z - (A_{11} + B_1K_1) & -(A_{12} + B_1K_2) \\ -(A_{21} + B_2K_1) & I_{n_2}z - (\overline{A}_{22} + B_2K_2) \end{bmatrix} =$$

$$= \begin{vmatrix} z+0.3 & 0 & -0.2542 & -0.0658 \\ 0 & z+0.04 & 0.0229 & -0.0171 \\ 0 & -0.05 & z+0.4687 & -0.1487 \\ 0 & 0 & 0.0458 & z+0.0658 \end{vmatrix}$$

$$= z^4 + 0.8744z^3 + 0.2166z^2 + 0.0141z + 0.0003$$

has positive coefficients.

Example 7. Given the positive fractional 2D Roesser model with $\alpha = 0.4$, $\beta = 0.9$ and

$$\begin{gathered} A_{11} = \begin{bmatrix} -0.4 & 0.01 \\ 0.03 & 0.001 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} 0.01 & 0.2 \\ 0 & 0.01 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.9 & 0.01 \\ 0.01 & -0.8 \end{bmatrix}, \\ B_1 = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.002 \end{bmatrix}. \end{gathered} \tag{129}$$

Find a gain matrix $K = [\ K_1 \quad K_2\ ]$, $K_i \in \Re^{1\times 2}$, $i = 1,2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (129) is unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.01 & 0.01 & 0.01 \\ 0.03 & 0.001 & 0.01 & 0.2 \\ 0.01 & 0.2 & -0.9 & 0.01 \\ 0 & 0.01 & 0.01 & -0.8 \end{bmatrix} \tag{130}$$

has positive diagonal entries.

We chose

$$\begin{gathered} D = [\ D_1 \quad D_2\ ], \qquad D_1 = [\ 0.13 \quad -0.37\ ], \\ D_2 = [\ -3.19 \quad -0.11\ ]. \end{gathered} \tag{131}$$

Applying Theorem 30 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (120) we obtain

$$\begin{gathered} \Lambda = blockdiag\,[\Lambda_1, \Lambda_2], \\ \Lambda_1 = \begin{bmatrix} 0.0554 & 0 \\ 0 & 0.0755 \end{bmatrix}, \\ \Lambda_2 = \begin{bmatrix} 0.8659 & 0 \\ 0 & 0.0032 \end{bmatrix}. \end{gathered} \tag{132}$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (116) we obtain the gain matrix

$$\begin{aligned} K &= [\ K_1 \quad K_2\ ] = [\ D_1\Lambda_1^{-1} \quad D_2\Lambda_2^{-1}\ ] = \\ &= [\ 2.3460 \quad -4.9035 \quad -3.6840 \quad -34.1058\ ]. \end{aligned} \tag{133}$$

The closed loop system is positive since matrices

$$\overline{A}_{11} + B_1K_1 = \begin{bmatrix} 0 & 0.01 \\ 0.0323 & 0.3961 \end{bmatrix},$$

$$A_{12} + B_1K_2 = \begin{bmatrix} 0.01 & 0.01 \\ 0.0063 & 0.1659 \end{bmatrix},$$

$$A_{21} + B_2K_1 = \begin{bmatrix} 0.01 & 0.2 \\ 0.0047 & 0.0002 \end{bmatrix},$$

$$\overline{A}_{22} + B_2K_2 = \begin{bmatrix} 0 & 0.01 \\ 0.0026 & 0.0318 \end{bmatrix}$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\det \begin{bmatrix} I_{n_1}z - (A_{11} + B_1K_1) & -(A_{12} + B_1K_2) \\ -(A_{21} + B_2K_1) & I_{n_2}z - (\overline{A}_{22} + B_2K_2) \end{bmatrix} =$$

$$= \begin{vmatrix} z+0.4 & -0.01 & -0.01 & -0.01 \\ -0.0323 & z+0.0039 & -0.0063 & -0.1659 \\ -0.01 & -0.2 & z+0.9 & -0.01 \\ -0.0047 & -0.0002 & -0.0026 & z+0.8682 \end{vmatrix}$$

$$= z^4 + 2.1721z^3 + 1.4953z^2 + 0.3159z + 0.0004$$

has positive coefficients.

5. Concluding remarks

The concepts of the practical stability and of the asymptotic stability of the positive fractional and the cone fractional discrete-time linear systems have been introduced. Necessary and sufficient conditions for the stabilities of the fractional systems have been established. It has been shown that the checking of the stabilities of positive 2D linear systems can be reduced to testing the stabilities of corresponding 1D positive linear systems. Three LMI approaches have been proposed for checking the stabilities of the positive fractional linear systems. The LMI approach has been applied to compute gain matrices of the state-feedbacks for the fractional 2D Roesser model such that the closed-loop systems are positive and asymptotically stable. Necessary and sufficient conditions for the solvability of the problem have been established. The effectiveness of the proposed LMI method has been demonstrated on numerical examples of the fractional 2D Roesser model. The considerations can be easily extended for positive fractional linear 1D and 2D systems with delays. An extension of these considerations for continuous-time 1D and 2D positive fractional linear systems is an open problem.

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