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Repairable systems availability optimization under imperfect maintenance

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Abstract. This paper deals with the modeling of a preventive maintenance strategy applied to a single-unit system subject to random failures. According to this policy, the system is subjected to imperfect periodic preventive maintenance restoring it to 'as good as new' with probability p and leaving it at state 'as bad as old' with probability q. Imperfect repairs are performed following failures occurring between consecutive preventive maintenance actions, i.e the times between failures follow a decreasing quasi-renewal process with parameter a. Considering the average durations of the preventive and corrective maintenance actions as well as their respective efficiency extents, a mathematical model is developed in order to study the evolution of the system stationary availability and determine the optimal PM period which maximizes it. The modeling of the imperfection of the corrective maintenance actions requires the knowledge of the quasi-renewal function. A new expression approximating this function is proposed for systems whose times to first failure follow a Gamma distribution. Numerical results are obtained and discussed.

Key words: maintenance strategies, imperfect maintenance, quasi-renewal process, gamma distribution.

1. Introduction

In the current context of an increasingly wild competition, the constraints of time, quality and cost imposed on the industrial companies put them in front of the obligation to ensure a maximum availability of their production equipment at the lowest cost. Under these conditions, the implementation of preventive maintenance strategies proves to be inevitable.

A maintenance strategy is defined as a decision rule which establishes the sequence of actions to be undertaken according to the state of the system. With each maintenance action one associates, a cost, a duration and a certain quality or efficiency. The performance of a strategy is then generally evaluated in terms of the average total cost on a given horizon or in terms of the system stationary availability. Other performance criteria are proposed in the literature.

A great number of publications propose several maintenance policies, implying different types of preventive and corrective actions, such as inspections, replacements by new or used identical equipment, overhauls or repair to the as good as new state, the minimal repair which consists in bringing the system back to the same operating state as just before failure (as bad as old state), etc.

Basic maintenance strategies have been proposed in [1]. A great number of publications on the subject followed with maintenance policies based on complex mathematical models using the renewal theory and many other stochastic processes. There are several reviews of the literature on the subject summarizing the various models and gathering them in different classes. Among these reviews, we find those in [2–3].

Various maintenance models consider that maintenance actions are perfectly executed. Actually, the effectiveness of maintenance actions is generally between the two extreme limits ('as good as new' and 'as bad as old'), what is generally called imperfect maintenance. Several models of imperfect maintenance were proposed in the literature. They can be classified in two categories: models based on an arithmetic reduction of the age of the system [4–5], and models of reduction of the intensity function [6–7].

With regard to the reduction of age, the improvement of the state of the system, following a maintenance action, is equivalent to an arithmetic reduction of its age. For the models of reduction of the intensity of failures, the improvement of the state of the system, following a maintenance action, is equivalent to a reduction of its failure rate of a quantity proportional to its value right before maintenance. A literature review on imperfect maintenance can be found in [8–9].

In this paper, we consider a general industrial framework where preventive and corrective maintenance actions are imperfect. In fact, we do not always find the best qualified technicians nor the most suitable tools or spare parts to carry out maintenance actions. We consider in this context a periodic preventive maintenance strategy applied to a single-unit system prone to random failures. A similar maintenance policy has been studied by Wang and Pham [8] considering cost optimization in a context of maintenance actions with negligible durations. Here, we model and optimize the system stationary availability for this maintenance policy taking into account the durations of maintenance actions. Moreover, due to the

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computation complexity of the quasi-renewal function used in the mathematical model, we propose a new expression of this function for systems whose times to first failure follow a Gamma distribution.

In next section we define the strategy and establish the expression of the system stationary availability. We also prove that there always exists an optimal preventive maintenance period, T^*, maximizing the system stationary availability for any given set of parameters regarding the maintenance actions duration and the system failure distribution. The system asymptotic average availability calls upon the quasi-renewal function whose analytical expression is very difficult, even impossible, to obtain in a closed form. In Sec. 3 we derive a new expression of the quasi-renewal function in the particular case of a Gamma distribution and we discuss the related numerical calculations. A numerical example is presented in Sec. 4. The obtained results will be presented and discussed. The paper is concluded in Sec. 6.

2. Strategy definition and mathematical model

We consider a randomly failing system with a known continuous lifetime probability distribution. An imperfect preventive maintenance action is periodically performed (every T time units). It follows a (p, q) rule, which means that it restores the system to the 'as good as new' state with probability p and it keeps it in the 'as bad as old' state with probability q ($q = 1 - p$). If the system fails between successive preventive maintenance actions, it undergoes an imperfect repair action after which the system failure inter-arrival time reduces to a fraction of its immediate previous one (all successive lifetimes being independent). Lifetimes follow a decreasing quasi-renewal process with parameter a [8, 10].

The preventive and corrective maintenance actions have respectively constant average durations T_p and T_c during which the system is unavailable.

We develop, in what follows, the expression of the system stationary availability under these conditions. This expression will allow finding the optimal period T^* for undertaking a preventive maintenance so as to maximize the system stationary availability. We will first recall the definition of the quasi-renewal process.

2.1. Definition of the quasi renewal process. Wang and Pham [8, 10] define the quasi-renewal process as follows: observe the sequence of non-negative random variables $\{T_1, T_2, ..., T_n\}$ (Fig. 1), the counting process $\{N(t), t > 0\}$ is said to be a quasi-renewal process with parameter a and the first inter-arrival time T_1, if $T_1 = Z_1$; $T_2 = a\,Z_2$; $T_3 = a^2\,Z_3$;...; $T_n = a^{n-1}\,Z_n$, where the Z_i are independent and identically distributed and $a > 0$ is a constant.

Since we consider imperfect repair throughout the paper, the constant 'a' will be taken between 0 and 1 ($0 < a < 1$). It is considered as a characteristic of the corrective maintenance actions efficiency. When $a = 1$, this process corresponds to the ordinary renewal process [1] with perfect repairs.

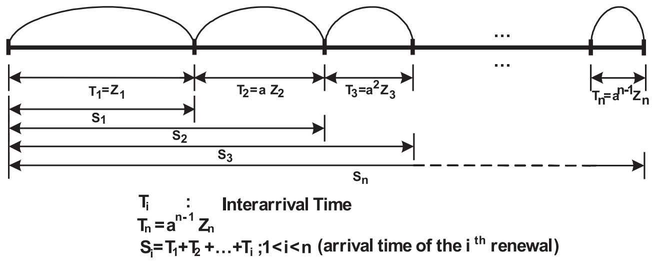

Fig. 1. The quasi-renewal process model

2.2. The system stationary availability model. Figure 2 illustrates the proposed maintenance policy as defined in the beginning of this section.

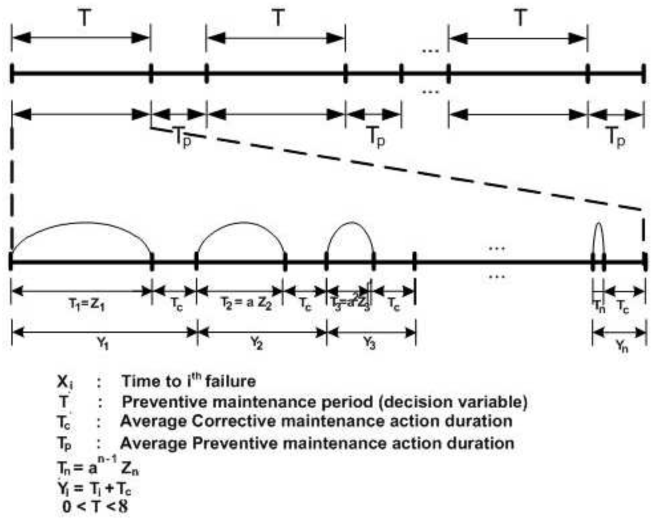

Fig. 2. The proposed maintenance strategy

The preventive maintenance period T is the decision variable. Let $SA(T)$ stand for the system stationary availability. Let's consider $D(T)$ as the average duration of a renewal cycle (period between successive perfect preventive maintenance actions), and $I(T)$ as the average period during which the system is unavailable (submitted to maintenance actions) during a renewal cycle.

Based on the classical renewal reward theory [1], the system stationary availability, $SA(T)$, can be expressed as follows:

$$SA(T) = 1 - \frac{I(T)}{D(T)}, \tag{1}$$

with

$$D(T) = \sum_{i=1}^{\infty} p\,q^{(i-1)} i\,(T + T_p), \tag{2}$$

$$I(T) = \sum_{i=1}^{\infty} p\,q^{(i-1)}\,(i\,T_p + T_c\,Q(iT)), \tag{3}$$

where $Q(T)$ stands for the average number of system restarts following failure during period T. It is called the quasi-renewal function.

Considering $F_{T_i}(.)$ as the probability distribution of the system times to the i^{th} failure T_i ($i = 1, 2, \ldots, n$), [8, 10] expressed the quasi-renewal function as follows:

$$Q(t) = \sum_{n=1}^{\infty} \overset{n}{\underset{i=1}{\Upsilon}} F_{T_i}(t)\,, \tag{4}$$

where $\overset{n}{\underset{i=1}{\Upsilon}} F_{T_i}(t)$ represents the convolution product of the inter-arrival times distributions $F_{T_i}(.)$; $(i = 1, 2, \ldots, n)$; and $F_{T_i}(t) = F_{T_1}\left(\frac{1}{a^{i-1}}t\right)$.

Hence, the quasi-renewal function can be determined knowing the system time to first failure distribution and the parameter a. Statistical estimation of parameter a is discussed in [9].

Combining Eqs. 1 to 3, we have:

$$1 - SA(T) = \frac{\sum_{i=1}^{\infty} p\, q^{(i-1)} \left(i\, T_p + T_c\, Q(iT)\right)}{\sum_{i=1}^{\infty} p\, q^{(i-1)} i\, (T + T_p)}.$$

Considering that:

$$\sum_{i=1}^{\infty} q^{(i-1)} i \; = \left(\frac{1}{1-q}\right)^2 = \frac{1}{p^2},$$

we have

$$1 - SA(T) = \frac{T_p + p^2\, T_c \sum_{i=1}^{\infty} q^{(i-1)}\; Q(iT)}{(T + T_p)}.$$

Hence, for any given set of known parameters T_p, T_c, a, and F_{T_1}, the system stationary availability model is as follows:

$$SA(T) = 1 - \frac{T_p + p^2 T_c \left(Q(T) + \sum_{i=2}^{\infty} q^{(i-1)} Q(iT)\right)}{(T + T_p)}. \tag{5}$$

Theorem.

The considered maintenance policy admits, for any given set of input parameters, F_{T_1}, a, T_p and T_c, a finite optimal period T^* of preventive maintenance which maximizes the system stationary availability.

Proof.

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Note that *SA*(T) is a continuous function for $0 < T < \infty$ since $Q(T)$ is continuous given that it is assumed that the system time to first failure distribution F_{T_1} is continuous
It is easy to see that:

$$SA(0) = 0.$$

On the other hand, we show in the appendix that: $\lim_{T\to\infty} SA(T) = 0$.

Consequently, we can conclude that there exists a finite period T^* which maximizes *SA*(T) for any given set of parameters: F_{T_1}, a, T_p and T_c.

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Curve 1 in Fig. 3 shows the general behaviour of the system stationary availability according to the preventive maintenance period T. Curves 2 and 3, in the same figure, show both possible behaviours of the system stationary availability in the case of perfect corrective and preventive maintenance actions (the block replacement policy [1]). It is interesting to notice that if an infinite optimal solution (no preventive maintenance – curve 2) is possible in the case of perfect maintenance, it can not be envisaged in an imperfect maintenance situation.

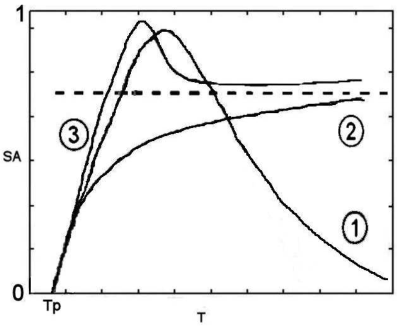

Fig. 3. General behaviours of system stationary availability function SA(T) in the situation of perfect maintenance (curves 2 and 3) and imperfect maintenance (curve 1)

Most of imperfect maintenance policies based on the quasi-renewal process, including the one considered in this paper, are modelled using the quasi-renewal function $Q(.)$. It is not possible to obtain this function expressed by Eq. (4) in closed form. One approximation in the case of the Normal distribution is given in [8] and [13] provides numerical approximations of the quasi-renewal function in the cases of exponential and Gamma distributions. These approximations are very difficult to obtain, they require the inversion of a truncated infinite sum in the Laplace transform space. In next section, we develop a new expression approximating the quasi-renewal function in the particular case of systems whose time to first failure follows a Gamma distribution.

3. Approximation of the quasi-renewal function in the case of a gamma distribution

We develop in what follows a new expression which approximates the quasi-renewal function for systems whose time to first failure follows a Gamma distribution with shape parameter α and scale parameter β.

$$f_{T_1}(t) = \frac{1}{\beta^\alpha\, \Gamma(\alpha)} t^{\alpha-1} \exp\left(-\frac{t}{\beta}\right), \tag{6}$$

with $\Gamma(\alpha) = \int_0^\infty \mathrm{x}^{\alpha-1}\mathrm{e}^{-x} dx$.

The quasi-renewal function associated with this system is given by:

$$Q(t) = \sum_{n=1}^{\infty} n\; \Pr\left[N(t) = n\right], \tag{7}$$

where $N(t)$ is a random variable representing the number of system restarts following failure during a time interval $[0, t]$. Given that $S_n = \sum_{i=1}^{n} T_i = Z_1 + a\, Z_2 + a^2 Z_3 + ... + a^{n-1} Z_n$ (see Fig. 1), we have:

$$\Pr\left[N(t) = n\right] = \Pr\left[S_{n+1} > t\right] - \Pr\left[S_n > t\right].$$

Let $G_{S_n}(t)$ be the probability distribution function associated to the random variable S_n. Hence,

$$\begin{aligned}\Pr\left[N(t)=n\right]&=\left[1-G_{S_{n+1}}(t)\right]-\left[1-G_{S_n}(t)\right]\\ \Pr\left[N(t)=n\right]&=G_{S_n}(t)-G_{S_{n+1}}(t).\end{aligned}\qquad(8)$$

Let's find an approximation of $G_{S_n}(t)$.

In the general case, [8] showed that:

$$f_{Ti}(t)=\frac{1}{a^{i-1}}f_{T_1}\left(\frac{t}{a^{i-1}}\right)\qquad\text{for}\qquad i=1,2,\ldots,n.$$

Thus, if the system times to first failure T_1 follow a Gamma distribution: $Gamma(\alpha,\ \beta)$, it is clear that the times T_i to the i^{th} failure follow a Gamma distribution: $Gamma(\alpha,\ \beta\, a^{i-1})$.

Given that the $Z_i(i=1,2,...,n)$ are independent, S_n is the sum of n independent $T_i(i=1,2,...,n)$, which means the sum of n independent random variables respectively distributed according to the Gamma law: $Gamma(\alpha,\ \beta\, a^{i-1})$.

The distribution of this sum of random variables, each one distributed according to a Gamma law, can be validly approached by a Gamma distribution [11–12].

Let $G_{S_n}(\Phi,\Psi)$ be this distribution function of S_n: a Gamma distribution with shape parameter Φ and scale parameter Ψ.

We use in what follows the two first moments of $G_{S_n}(\Phi,\Psi)$ to determine its two parameters Φ et Ψ.
The first moment:

$$\begin{aligned}E\left[S_n\right]&=E\left[T_1+T_2+T_3+...+T_n\right]\\&=E\left[Z_1\right]+a\,E\left[Z_2\right]+a^2E\left[Z_3\right]+...+a^{n-1}E\left[Z_n\right],\end{aligned}$$

with $E[Z_i]=\alpha\,\beta$ for $i=1,2,...,n$ and $0<a\ <1$

$$E[S_n]=\alpha\,\beta+a\,\alpha\,\beta+...+a^{n-1}\alpha\,\beta=\frac{1-a^n}{1-a}\alpha\,\beta.$$

Then we consider the variance:

$$\begin{aligned}Var\left[S_n\right]&=Var\left[T_1+T_2+T_3+...+T_n\right]\\&=Var\left[Z_1+a\,Z_2+a^2Z_3+...+a^{n-1}Z_n\right].\end{aligned}$$

The $Z_i(i=1,2,...,n)$ being independent, we can write:

$$Var[S_n]=Var[Z_1]+a^2\,Var[Z_2]+...+a^{2(n-1)}Var[Z_n],$$

with $Var[Z_i]=\alpha\,\beta^2$ for $i=1,2,...,n$ and $0<a\ <1$

$$Var[S_n]=\alpha\,\beta^2+...+a^{2(n-1)}\alpha\,\beta^2=\frac{1-a^{2n}}{1-a^2}\alpha\,\beta^2.$$

Given, by definition of the Gamma distribution, that $E[S_n]=\Psi\,\Phi$ and $Var[S_n]=\Psi^2\,\Phi$, we obtain by identification :

$$\left\{\begin{aligned}\Psi\,\Phi&=\frac{1-a^n}{1-a}\,\alpha\,\beta\\ \Psi^2\,\Phi&=\frac{1-a^{2n}}{1-a^2}\,\alpha\,\beta^2\end{aligned}\right..$$

Solving this system leads to the following result:

$$\Phi=\frac{1-a^n}{1-a}\frac{1+a}{1+a^n}\alpha$$

and

$$\Psi=\frac{1+a^n}{1+a}\beta.$$

The probability distribution associated with S_n is then a Gamma distribution defined as follows:

$$G_{S_n}(\Phi,\Psi);\quad 0<a<1.\qquad(9)$$

Hence, returning to equations (7) and (8), the quasi-renewal function in case the first times to failure follow a Gamma distribution can be expressed as follows:

$$Q(t)=\sum_{n=1}^{\infty}n\left[G_{S_n}(t)-G_{S_{n+1}}(t)\right]=\sum_{n=1}^{\infty}G_{S_n}(t),\qquad(10)$$

with $G_{S_n}(t)$ being the Gamma distribution given by Eq. (9).

It can easily be shown that for the case of a perfect renewal process ($a=1$) [1] with systems whose times to failure follow a Gamma distribution $G(\alpha,\beta)$, the renewal function is given by:

$$M(t)=\sum_{n=1}^{\infty}Gamma(n\,\alpha,\beta).\qquad(11)$$

The formulation of the quasi-renewal function as a sum of distribution functions (Eq. 10), does not raise any special programming or computation problem. This in opposition to the formulation proposed by Rehmert [13] which requires an inversion of a truncated infinite sum in the Laplace transform space which is very difficult and sometimes impossible to perform.

3.1. Numerical calculation of the quasi-renewal function. For systems whose times to first failure are distributed according to Gamma distributions, the quasi-renewal function (Eq. 10) is an infinite sum of probability distribution functions. Each term of the sum represents the probability to have n imperfect repairs (quasi-renewals) in a time interval $[0,t]$. The computer programming of this quasi-renewal function requires truncating the sum at a certain level $n=c$.

$$Q(t)=\sum_{n=1}^{c}G_{S_n}(t).\qquad(12)$$

The choice of the value of c depends on the working horizon of interest t.

For illustration purpose, let us consider the case where the first time to failure follows a Gamma distribution with shape parameter $\alpha=3$ and scale parameter$\beta=3$, with a corrective maintenance action efficiency factor $a=0.9$. Figure 4 shows the curves of the first three terms (distribution functions) of Eq. (12) as well as the curve of the sum of these three functions in case $c=3$. Figure 5 illustrates the same thing for $c=5$.

Notice that truncating the sum (Eq. 12) at $c=3$ allows determining the quasi-renewal function for time periods t lower or equal than approximately $t_c=32$ time units. In case one would like to evaluate the quasi-renewal function for longer periods, truncation at higher values of c will be needed.

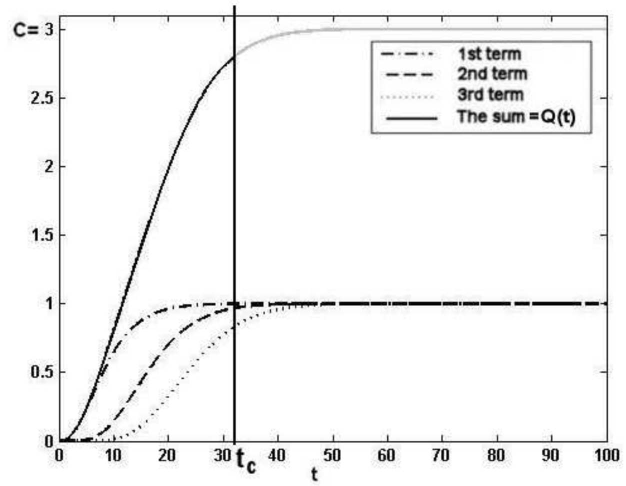

Fig. 4. The terms of the quasi-renewal function and their sum for $c = 3$ (Eq. 12)

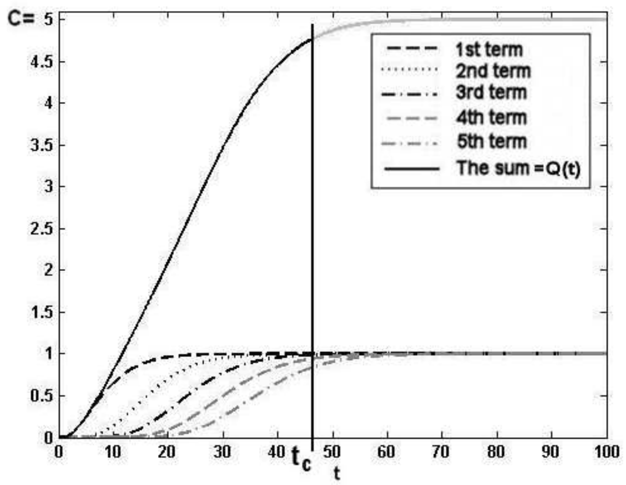

Fig. 5. The terms of the quasi-renewal function and their sum for $c = 5$ (Eq. 12)

Indeed, as shown below in Fig. 5, truncation at $c = 5$, allows evaluating the quasi-renewal function over a larger time interval: [0, 46] time units.

4. Numerical example

For illustration purpose of the considered maintenance policy, let's consider a randomly failing system whose time to first failure follows a Gamma distribution with shape parameter $\alpha = 3$ and scale parameter $\beta = 3$.

The following average durations are considered:

– Average duration of the corrective maintenance actions: $T_c = 0.35$ time unit;
– Average duration of the preventive maintenance actions: $T_p = 0.15$ time unit.

The corrective maintenance actions are characterized by a constant efficiency factor $a = 0.6$

For different values p of the probability to have perfect preventive maintenance actions, Table 1 shows the optimal preventive maintenance periods T^* which maximize the system availability $SA(T)$. As it should be expected, preventive maintenance actions have to be performed more frequently (T^* decreases) as their efficiency extent decreases.

Table 1
Optimal policies for different preventive maintenance efficiency extents

p	T^*	$SA^*(T)$
1	5.59	0.9545
0.8	4	0.9413
0.6	2.95	0.9239
0.4	2.1	0.8973
0.2	1.4	0.8550

Figure 6 shows the evolution of the system stationary availability $SA(T)$ for different values of p considering a truncation in the quasi-renewal function at $c = 50$ (Eq. 12).

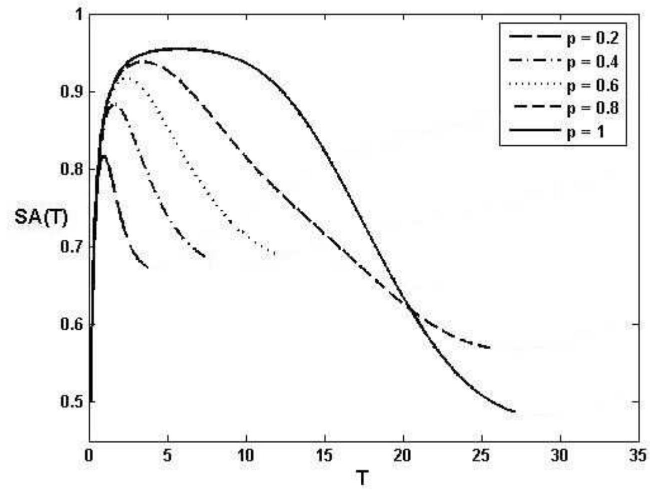

Fig. 6. Evolution of the system stationary availability $SA(T)$ with different preventive maintenance factors p

Notice that as preventive maintenance actions get more efficient (increasing p), the optimal values of T^* get larger (PM actions should be performed less frequently). Moreover, efficient PM actions allow more flexibility when implementing the optimal policy in practice. Indeed, as it is clearly shown by the availability curves, a given deviation from the optimal solution (in terms of PM period T) yields significantly less availability loss as PM actions get more efficient.

Considering the same input parameters, Fig. 7 shows that for a given efficiency degree p of preventive maintenance, as the corrective maintenance actions become more and more efficient (increasing a), the optimal preventive maintenance period T^* becomes larger and the corresponding system availability $SA(T^*)$ increases. It is also interesting to notice that beyond a certain level of the repair efficiency factor a, the optimal availability variations tend to stabilize. This allows concluding that it is possible to tolerate a certain degree of ineffectiveness of the corrective actions without having a notable availability loss. This flexibility increases as preventive maintenance gets more efficient (increasing p). This figure clearly shows how preventive and corrective maintenance efficiency extents affect simultaneously the system optimal availability.

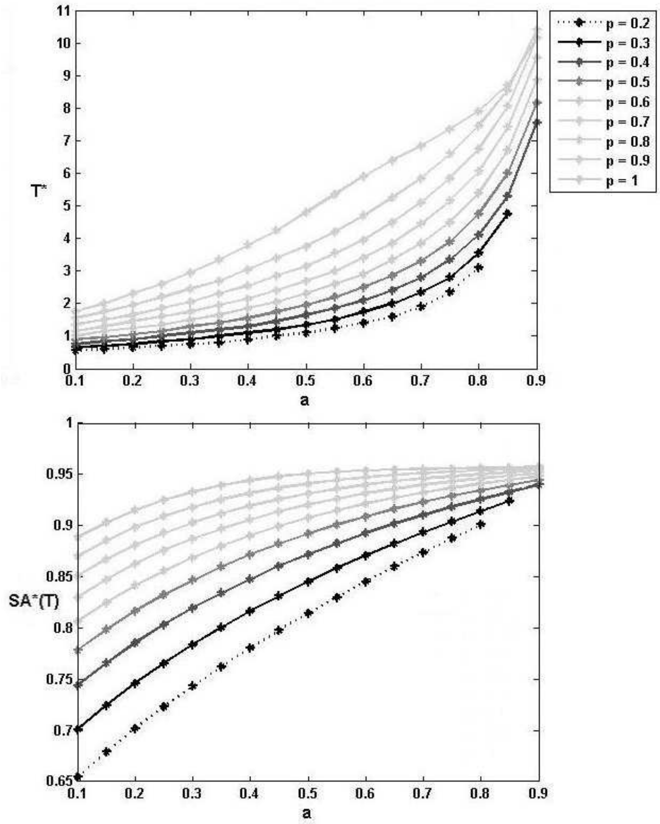

Fig. 7. T^* and *SA*(T^*) evolution according to the repairs efficiency factor, with $T_c = 0.35$ and $T_p = 0.15$

For the same system and for $p = 0.7$, Fig. 8 displays optimal strategies in terms of the repair efficiency factor a for different ratios T_c/T_p. We notice that for any given repair efficiency degree a, as the ratio T_c/T_p increases, the optimal period of preventive maintenance T^* and the corresponding optimal stationary availability decrease.

Finally, Fig. 9 shows the evolution of the optimal policy according to the shape parameter α of the Gamma distribution associated with the first times to failure of the system. We notice essentially that for all the considered systems, the optimal availability level increases as the repairs are considered more and more efficient. This is accomplished until reaching a relative stability from a certain repair efficiency factor level (approximately equal to 0.6 for $\alpha = 5$ for example). Here again, the following important conclusion can be drawn: for several systems with different failure rates, the degree of the repair efficiency stops affecting significantly the optimal availability from a certain threshold which increases as the system is less reliable (lower shape parameter).

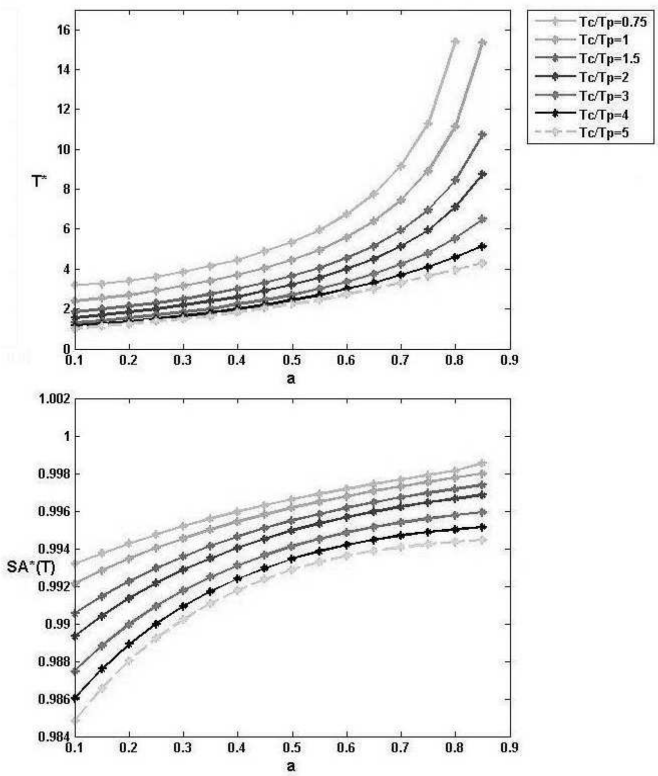

Fig. 8. T^* and *SA*(T^*) evolution according to the corrective repair efficiency factor, for various ratios T_c/T_p and a fixed preventive maintenance factor $p = 0.7$

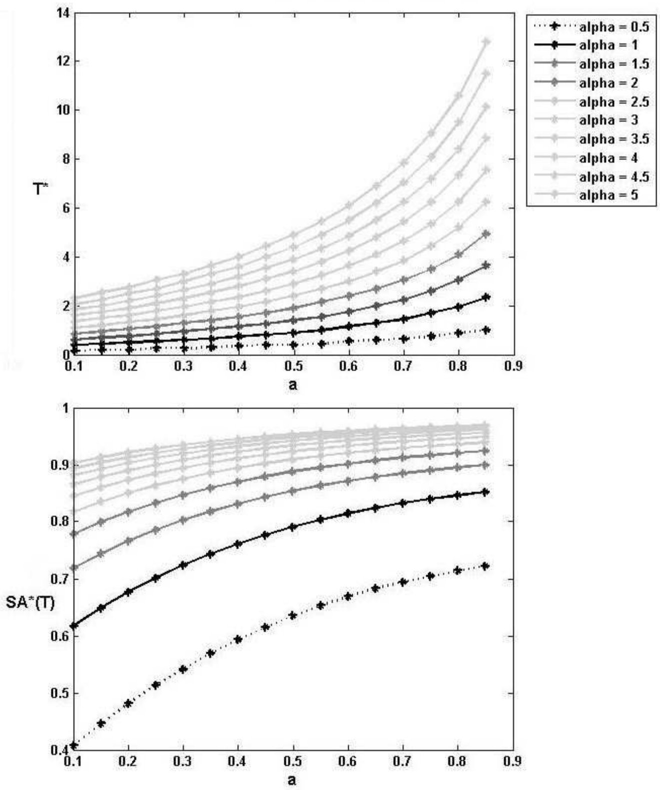

Fig. 9. T^* and *SA*(T^*) evolution according to the preventive action efficiency factor, for different Gamma distribution shape parameters with fixed preventive efficiency factor $p = 0.7$

5. Conclusions

In this work, we studied a preventive maintenance policy for randomly failing systems evolving in a context where preventive and corrective maintenance actions are imperfect. The imperfection of repair actions has been modelled by a decreasing quasi-renewal process based on a deterministic repair efficiency factor a. Preventive maintenance actions follow a (p, q) rule. We modelled the system stationary availability and studied its evolution under the proposed maintenance policy. We considered the general case and the particular situations of systems whose times to first failure follow a Gamma distribution. We derived for such systems a new expression to approximate and easily compute the quasi-renewal function.

This study showed that for any given situation regarding the system, the repair and preventive maintenance efficiency extents, and the downtime durations for preventive and corrective maintenance, there is necessarily a finite optimal period T^* of preventive maintenance which maximizes the system stationary availability.

Obtained numerical results corresponding to a set of particular cases, illustrated how preventive and corrective maintenance efficiency extents affect simultaneously the system optimal availability. It allowed concluding that as preventive maintenance gets more efficient, it is possible to tolerate a certain degree of ineffectiveness of the corrective actions without having an important availability loss. The knowledge of the interval of tolerance of the repair inefficiency degree not causing significant availability loss represents an important source of flexibility for the maintenance management.

Finally, the obtained results also showed that for several systems with different failure rates, the degree of the repair efficiency stops affecting significantly the optimal availability from a certain threshold level which increases as the system is less reliable.

Several extensions of this work considering more complex imperfect maintenance policies are currently under consideration. Also, the quasi-renewal function being a key quantity for a great number of imperfect maintenance models based on the quasi-renewal process, we are working on the development of a general numerical algorithm which allows the computation of the quasi-renewal function for any given distribution of systems time to first failure.

Appendix

Proof of $\lim\limits_{T\to\infty} SA(T) = 0$

$$\lim_{T\to\infty} SA(T) = \lim_{T\to\infty} 1 - \frac{T_p + p^2\, T_c \sum\limits_{i=1}^{\infty} q^{(i-1)}\, Q(iT)}{(T+T_p)},$$

$$\lim_{T\to\infty} SA(T) = 1 - T_c\, p^2 \left[\sum_{i=1}^{\infty} q^{(i-1)} \lim_{T\to\infty} \frac{Q(iT)}{(T+T_p)} \right] \quad \text{(A1)}$$

$$\lim_{T\to\infty} \frac{Q(iT)}{(T+T_p)} = \lim_{T\to\infty} \frac{Q(iT)}{T}.$$

Hence, from Eq. (A1) above:

$$\lim_{T\to\infty} SA(T) = 1 - T_c\, p^2 \left[\sum_{i=1}^{\infty} q^{(i-1)} \lim_{T\to\infty} \frac{Q(iT)}{T} \right],$$

$$\lim_{T\to\infty} SA(T) = 1 - T_c\, p^2 \left[\sum_{i=1}^{\infty} q^{(i-1)} i \lim_{T\to\infty} \frac{Q(iT)}{iT} \right]. \quad \text{(A2)}$$

The expression $\frac{Q(T)}{T}$ represents the average number of restarts per time unit in the time interval $[0, T]$. We can also consider this term as the inverse of the average period between successive restarts over the same interval $[0, T]$. Thus, according to the quasi-renewal process, defined in Subsec. 2.1, and considering Fig. 2:

- T_i: the system time to the i^{th} failure following a given probability distribution,
- Y_i: a random variable defined as the sum of T_i and T_c.

We have:

$$\lim_{T\to\infty} \frac{Q(T)}{T} = \lim_{n\to\infty} \left[\frac{1}{\sum\limits_{i=1}^{n} Y_i \Big/ n} \right] = \lim_{n\to\infty} \left[\frac{n}{\sum\limits_{i=1}^{n} (T_i + T_c)} \right],$$

$$\lim_{T\to\infty} \frac{Q(T)}{T} = \lim_{n\to\infty} \left[\frac{n}{\sum\limits_{i=1}^{n} T_i + n\, T_c} \right],$$

$$\lim_{T\to\infty} \frac{Q(T)}{T} = \lim_{n\to\infty} \left[\frac{n}{\sum\limits_{i=1}^{n} a^{i-1} Z_i + n\, T_c} \right].$$

Note that $\sum\limits_{i=1}^{n} a^{i-1} Z_i$ is finite because $\lim\limits_{n\to\infty} a^{n-1} Z_n = 0$ for $0 < a < 1$. Hence,

$$\lim_{T\to\infty} \frac{Q(T)}{T} = \frac{1}{T_c}.$$

Let's define $\xi = iT$; when $T \to \infty$, $\xi \to \infty$; consequently:

$$\lim_{\xi\to\infty} \frac{Q(\xi)}{\xi} = \frac{1}{T_c}.$$

Going back to Eq. (A2), we obtain:

$$\lim_{T\to\infty} SA(T) = 1 - T_c\, p^2 \left[\sum_{i=1}^{\infty} q^{(i-1)} i \frac{1}{T_c} \right].$$

Given that

$$\sum_{i=1}^{\infty} q^{(i-1)} i = \left(\frac{1}{1-q} \right)^2 = \frac{1}{p^2},$$

we have:

$$\lim_{T\to\infty} SA(T) = 1 - T_c\, p^2 \frac{1}{p^2 T_c} = 0,$$

for $0 < a < 1$.

Hence, it is proved that:

$$\lim_{T\to\infty} SA(T) = 0.$$

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