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# A new method for analytic determination of extremum of the transients in linear systems

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**Abstract.** The relation between extremal values of the error and the coefficients of its differential equations is one of the central problems of control systems in chemical industry, becouse extremal values of the error sometimes cause serious damages to the environment or to the system itself. Analytical formulae for the determination of these values are known only for the second-order systems. In this paper a method which permits to determine extremal values of the error in higher-order systems is proposed.

**Key words:** transcendental equations, extremal dynamic error, linear stationary system, parametric optimization, analytic formulae, discriminants of exponential functions, Vandermonde's determinant, Viéte's formulae, process control.

## 1. Introduction

In the process of design of the dynamic control systems we encounter the problem of determining the maximal transient error $x_e$ and the moment of time $t_e$ when it appears. The maximal error $x_e$ characteries the attainable accuracy, and time $t_e$ – the velocity of the rise of the transients [1, 2]. Let us consider the differential equation determining the transient error in linear control system of the $n$-th order with lumped and constant parameters:

$$a_0\frac{d^n x(t)}{dt^n}+a_1\frac{d^{n-1}x(t)}{dt^{n-1}}+\ldots+a_{n-1}\frac{dx(t)}{dt}+a_n x(t)=0. \tag{1}$$

The initial conditions are determined by the force function and the system parameteres $a_0$, $a_1$, $a_2$, …, $a_n$.
Let us assume, in general that

$$x^{(i-1)}(0)=c_i\neq 0, \qquad \text{for} \qquad i=1,2,\ldots,n.$$

The solution of Eq. (1) takes the form

$$x(t)=\sum_{k=1}^{n}A_k e^{s_k t}, \tag{2}$$

where $s_k$ are the real, different roots of the characteristic equation

$$a_0 s^n+a_1 s^{n-1}+\ldots+a_{n-1}s+a_n=0. \tag{3}$$

We denote

$$x^{(1)}(t)=\frac{dx^{(1)}(t)}{dt^{(1)}}.$$

The necessary condition for the transient error $x(t)$ to attain an extremal value at $t=t_e$ is given by the relation

$$\frac{dx(t)}{dt}=\sum_{k=1}^{n}s_k A_k e^{s_k t}=0.$$

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We will also need higher derivatives and use the relations

$$\frac{d^p x(t)}{dt}=\sum_{k=1}^{n}s_k^p A_k e^{s_k t}, \quad p=1,2,\ldots,n-1. \tag{4}$$

The Eqs. (2) and (4) represent a system of $n$ linear equations with respect to unknown terms $A_k e^{s_k t}$. Its matrix of coefficients is the Vandermonde's matrix:

$$\begin{pmatrix} 1 & 1 & \ldots & 1\\ s_1 & s_2 & \ldots & s_n\\ \vdots & \vdots & & \vdots\\ s_1^{n-1} & s_2^{n-1} & \ldots & s_n^{n-1}\end{pmatrix}. \tag{5}$$

Without loss of generality we assume for the sake of simplicity that Eq. (3) has only single roots: $s_i\neq s_j$ for $i\neq j$. With this assumption the matrix (5) has an inverse and the system (2) and (4) can be solved.

For this purpose we denote by $V$ the Vandermonde's determinant of the matrix (5) and by $V_j$ the Vandermonde's determinant of order $(n-1)$ of the variables $s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_n$.

We denote also by $\varphi_r^{(j)}$ the fundamental symmetric function of the $r$-th order of $(n-1)$ variables $s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_n$; $r=0,1,\ldots,n-1$:

$$\left.\begin{aligned}\varphi_0^{(j)}&=1\\ \varphi_r^{(j)}&=\textstyle\sum_{i=0}^{r}(-1)^r a_{r-i}s_j^i,\ j=1,2,\ldots,n-1,\ a_0=1\end{aligned}\right\}. \tag{6}$$

It is possible to show that the elements of the inverse matrix to the matrix (5) have the form:

$$\alpha_{ij}=\frac{(-1)^{i+j}}{V}\cdot\varphi_{n-j}^{(i)}V_i.$$

The solution of the system (2) and (4) is as follows

$$A_k e^{s_k t} = \sum_{j=1}^{n} \alpha_{kj} x^{(j-1)}(t) = = \sum_{j=1}^{n} \frac{(-1)^{k+j}}{V} \cdot \varphi_{n-j}^{(k)} V_k x^{(j-1)}(t)$$

or

$$A_k e^{s_k t} = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \varphi_{n-j}^{(k)} x^{(j-1)}(t) \qquad k = 1, 2, \ldots, n. \tag{7}$$

Its is evident that for $t = 0$ we know $x^{(j-1)}(0)$, and the substitution of $t = 0$ into Eq.(7) gives:

$$A_k = \frac{(-1)^k V_k}{V} \sum_{j=1}^{n} (-1)^j \varphi_{n-j}^{(k)} x^{(j-1)}(0) \ ,$$

or in the explicit form

$$A_k = \left\{ \frac{c_n - \sum_{v=1, v\neq k}^{n} s_v c_{n-1} + \sum_{v=1, v\neq k}^{n} s_v s_k c_{n-2}}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)} \ldots \right. \left. \ldots \frac{+ \ldots + (-1)^{n-1} \prod\limits_{v=1, v\neq k}^{n} s_v c_1}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)} \right\}$$

for $k = 1, 2, \ldots, n$.

The solution of Eq. (2) in a form more convenient for our consideration is:

$$x(t) = \sum_{k=1}^{n} \left\{ \frac{c_n e^{s_k t} - c_{n-1} e^{s_k t} \sum_{v=1, v\neq k}^{n} s_v + c_{n-2} e^{s_k t}}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)} \ldots \right. \left. \ldots \frac{\sum_{v=1, v\neq k}^{n} s_v s_k + \ldots + (-1)^{n-1} c_1 e^{s_k t} \prod\limits_{v=1, v\neq k}^{n} s_v}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)} \right\}. \tag{8}$$

It is worth noting that for particular $c_i$ we have symmetrical functions of $s_i$ without one $s_j$.

For the equation of order $n$ it is necessary to obtain the additional equations. For this purpose we assume that

$$\frac{dx(t)}{dc_i} = 0 \quad \text{or} \quad \frac{dx^{(1)}(t)}{dc_i} = 0 \quad i = 1, 2, \ldots, n. \tag{9}$$

The necessary number of such equations is $(n-2)$ and with the basic equation $x(t) = 0$, or $x^{(1)}(t) = 0$ this gives $(n-1)$ equations for determination of $e^{(s_n - s_i)t}$, $i = 1, 2, \ldots, n-1$ unknowns. We stress that the time $t$ must be positive, which means $0 \le t \le \infty$, and for maintaining asymptotic stability conditions it is required that $Re\ s_j < 0$.

According to this, the exponential functions $\underset{t=\infty}{0} \le e^{(s_n - s_i)t} \le \underset{t=0}{1}$, where $s_n < s_{n-1} < \ldots < s_2 < s_1 < 0$, $t \ge 0$.

This method is illustrated by an example of the equation of 3-rd order, which in general cannot be solved in analytical form.

## 2. Solution of the third order equation

Let us consider the equation [3]

$$\frac{d^3 x(t)}{dt^3} + a_1 \frac{d^2 x(t)}{dt^2} + a_2 \frac{dx(t)}{dt} + a_3 x(t) = 0. \tag{10}$$

with the initial conditions

$$x(0) = c_1, \quad x^{(1)}(0) = c_2, \quad x^{(2)}(0) = c_3$$

where $a_1$, $a_2$, $a_3$ are the constant coefficients.

The characteristic Eq. (10) is

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0. \tag{11}$$

We assume that the roots of Eq. (11) are different $s_i \neq s_j$ for $i, j = 1, 2, 3$. The solution of Eq. (10) is as follows:

$$x(t) = \frac{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1}{(s_1 - s_2)(s_1 - s_3)} \ e^{s_1 t} + + \frac{c_3 - (s_3 + s_1)c_2 + s_3 s_1 c_1}{(s_2 - s_3)(s_2 - s_1)} \ e^{s_2 t} + + \frac{c_3 - (s_1 + s_2)c_2 + s_1 s_2 c_1}{(s_3 - s_1)(s_3 - s_2)} \ e^{s_3 t}$$

or in a form more convenient for our purposes (8)

$$\begin{aligned} x(t) = {} & \frac{1}{(s_1 - s_3)(s_1 - s_2)(s_2 - s_3)} \cdot \\ & \Big\{ c_3 \Big[ (s_2 - s_3) e^{s_1 t} + (s_3 - s_1) e^{s_2 t} + (s_1 - s_2) e^{s_3 t} \Big] - \\ & - c_2 \Big[ (s_2^2 - s_3^2) e^{s_1 t} + (s_3^2 - s_1^2) e^{s_2 t} + (s_1^2 - s_2^2) e^{s_3 t} \Big] + \\ & + c_1 \Big[ s_2 s_3 (s_2 - s_3) e^{s_1 t} + s_3 s_1 (s_3 - s_1) e^{s_2 t} + \\ & + s_1 s_2 (s_1 - s_2) e^{s_3 t} \Big] \Big\}. \end{aligned} \tag{12}$$

Similarly, for the derivative $x^{(1)}(t)$ we have after differentiation (12)

$$\begin{aligned} x^{(1)}(t) = {} & \frac{1}{(s_1 - s_3)(s_1 - s_2)(s_2 - s_3)} \cdot \\ & \Big\{ c_3 \Big[ s_1 (s_2 - s_3) e^{s_1 t} + s_2 (s_3 - s_1) e^{s_2 t} + s_3 (s_1 - s_2) e^{s_3 t} \Big] - \\ & - c_2 \Big[ s_1 (s_2^2 - s_3^2) e^{s_1 t} + s_2 (s_3^2 - s_1^2) e^{s_2 t} + s_3 (s_1^2 - s_2^2) e^{s_3 t} \Big] + \\ & + c_1 s_1 s_2 s_3 \Big[ (s_2 - s_3) e^{s_1 t} + (s_3 - s_1) e^{s_2 t} + (s_1 - s_2) e^{s_3 t} \Big] \Big\}. \end{aligned} \tag{13}$$

We will use the Eq. (12) for finding zeroes of $x(t)$, and the Eq. (13) for determination of extremum points of $x(t)$. We start with determination of extremum points of $x(t)$. From the formulae (13) we obtain the necessary condition for extremum

$$x^{(1)}(t_e) = 0. \tag{14}$$

Equation (14) consists of three exponential terms and in general, it is not possible to obtain its solution analytically without an additional equation. The additional equation may be found in three variants, see (9)

1°

$$\frac{dx^{(1)}}{dc_3} = 0, \tag{15}$$

2°

$$\frac{dx^{(1)}}{dc_2} = 0, \tag{16}$$

3°

$$\frac{dx^{(1)}}{dc_1} = 0. \tag{17}$$

In the first variant we will use Eq. (15), then from the Eqs. (14) and (15) using the relation (13) we obtain that extremum points $t_1, t_2$ are:

$$e^{(s_1-s_3)t_1} = \frac{s_3}{s_1}\,\frac{c_2 - c_1 s_1}{c_2 - c_1 s_3}, \tag{18}$$

$$e^{(s_2-s_3)t_2} = \frac{s_3}{s_2}\,\frac{c_2 - c_1 s_2}{c_2 - c_1 s_3}. \tag{19}$$

In the particular case, if $c_1 = 0$ we have from (18), (19) that

$$\frac{e^{s_1 t_1}}{e^{s_3 t_1}} = \frac{s_3}{s_1}, \tag{20}$$

$$\frac{e^{s_2 t_2}}{e^{s_3 t_2}} = \frac{s_3}{s_2}\ . \tag{21}$$

If the initial condition $c_2 = 0$, we obtain that $t_1 = t_2 = 0$.
In the second variant using Eq. (16) we find that

$$e^{(s_1-s_3)t_1} = \frac{s_3}{s_1}\,\frac{c_3 - s_1^2 c_1}{c_3 - s_3^2 c_1}, \tag{22}$$

$$e^{(s_2-s_3)t_2} = \frac{s_3}{s_2}\,\frac{c_3 - s_2^2 c_1}{c_3 - s_3^2 c_1}. \tag{23}$$

If additionally $c_1 = 0$, then we obtain the relations (20) and (21) and if $c_3 = 0$ we receive from (22) and (23)

$$e^{(s_1-s_3)t_1} = \frac{s_1}{s_3}, \tag{24}$$

$$e^{(s_2-s_3)t_2} = \frac{s_2}{s_3}. \tag{25}$$

Equations (24) and (25) have no solutions for $t \geq 0$.
In the third variant we obtain using Eq. (17) that

$$e^{(s_1-s_3)t_1} = \frac{s_2 c_2 - s_1(s_2 - s_3)c_2}{(s_2 - s_3)(c_3 - s_3 c_2)},$$

$$e^{(s_2-s_3)t_2} = \frac{s_1 c_3 - s_2(s_3 - s_1)c_2}{(s_3 - s_1)(c_3 - s_3 c_2)}.$$

If additionally we assume $c_3 = 0$, then no $t_1 > 0$, $t_2 > 0$ exist and the same is for $c_2 = 0$.

In the next article a generalizatiom of the method and the numerical examples will be presented.

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