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Simple stability conditions for linear positive discrete-time systems with delays

M. BUSŁOWICZ*

Faculty of Electrical Engineering, Białystok Technical University, 45D Wiejska St., 15-351 Białystok, Poland

Abstract. Simple new necessary and sufficient conditions for asymptotic stability of the positive linear discrete-time systems with delays in states are established. It is shown that asymptotic stability of the system is equivalent to asymptotic stability of the corresponding positive discrete-time system without delays of the same size. The considerations are illustrated by numerical examples.

Key words: stability, robust stability, linear system, positive, discrete-time, delays, interval system.

1. Introduction

A dynamical system is called positive if any trajectory of the system starting form non-negative initial states remains forever non-negative for non-negative controls. An overview of state of the art in positive systems theory is given in the monographs [1, 2].

The problems of control and stability of systems with delays have been investigated in the monographs [3–7]. Recently, conditions for stability and robust stability of linear positive discrete-time systems with delays have been given in [8–18]. The problem of componentwise asymptotic stability and exponential stability of such systems is studied in [19].

In this paper simple new necessary and sufficient conditions for asymptotic stability of linear positive discrete-time systems with delays in states are given.

In the paper the following notations will be used: $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n = \Re_+^{n\times 1}$; Z_+ – the set of non-negative integers; I_n – the $n\times n$ identity matrix; a vector $x\in\Re^n$ will be called strictly positive (strictly negative) and denoted by $x>0$ ($x<0$) if all entries are positive (negative).

2. Preliminaries

Consider the positive discrete-time linear system with delays described by the homogeneous equation

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k}, \quad i\in Z_+, \tag{1}$$

with the initial condition

$$x_{-k}\in\Re_+^n \quad \text{for} \quad k=0,1,...,h, \tag{2}$$

where $x_i\in\Re^n$ is the state vector, $A_k\in\Re_+^{n\times n}$ ($k=0,1,...,h$) and h is a positive integer.
If

$$A_k\in\Re_+^{n\times n} \quad (k=0,1,...,h), \tag{3}$$

then $x_i\in\Re_+^n$ for all $i\in Z_+$ and for every $x_{-i}\in\Re_+^n$ ($i=0,1,...,h$) [17, 18].

The system (1) is asymptotically stable if and only if all roots of the characteristic equation

$$\det\left(zI_n - \sum_{k=0}^{h} A_k z^{-k}\right) = 0, \tag{4}$$

have moduli less than 1, or equivalently, all roots of the equation

$$\begin{aligned}\det&\left(z^{h+1}I_n - \sum_{k=0}^{h} A_k z^{h-k}\right)\\ &= z^{\tilde n} + a_{\tilde n-1}z^{\tilde n-1}+...+a_1 z + a_0 = 0,\end{aligned} \tag{5}$$

satisfy the condition $|z_k|<1$ for $k=1,2,...,\tilde n=(h+1)n$.

The positive system without delays equivalent to the positive system (1) has the form

$$\tilde x_{i+1} = \tilde A\tilde x_i, \quad i\in Z_+, \tag{6}$$

where $\tilde x_i = [x_i^{\mathrm T}, x_{i-1}^{\mathrm T}, \cdots, x_{i-h}^{\mathrm T}]^{\mathrm T}\in\Re_+^{\tilde n}$, $\tilde A\in\Re_+^{\tilde n\times\tilde n}$, $\tilde n=(h+1)n$ and

$$\tilde A = \begin{bmatrix} A_0 & A_1 & \cdots & A_{h-1} & A_h\\ I_n & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & I_n & 0\end{bmatrix}. \tag{7}$$

The positive system (6) is asymptotically stable if and only if all eigenvalues of the matrix $\tilde A$ have moduli less than 1.

In [17] it was shown that

$$\det(zI_{\tilde n}-\tilde A) = \det\left(z^{h+1}I_n - \sum_{k=0}^{h} A_k z^{h-k}\right). \tag{8}$$

This means that asymptotic stability of the system (1) (with delays) is equivalent to asymptotic stability of the system (6) (without delays).

*e-mail: busmiko@pb.edu.pl

Theorem 1 [17]. The positive system with delays (1) is asymptotically stable if and only if one of the following equivalent conditions holds

1) all coefficients of the characteristic polynomial of the matrix $\tilde{A} - I_{\tilde{n}}$ are positive,
2) all leading principal minors of the matrix $I_{\tilde{n}} - \tilde{A}$ are positive.

Based on Theorem 1, the necessary and sufficient conditions for asymptotic stability and robust stability of special class of the positive system (1) with delays have been given in [8–14].

The aim of this paper is to give simple new necessary and sufficient conditions for asymptotic stability of linear positive discrete-time systems with delays and for robust stability of interval positive systems with delays. First, we show that asymptotic stability of the positive system (1) is equivalent to asymptotic stability of the corresponding positive system without delays of the size n, i.e. of the size extremely less than the degree of the system (6) (equal to $\tilde{n} = (h+1)n$). Next, we generalise this result to the interval positive discrete-time systems with delays.

3. The main results

Consider the positive discrete-time linear system (1) with initial condition (2) satisfying the assumption: $x_{-k} > 0$ for at least one $k = 0, 1, ..., h$.

Theorem 2. The positive system (1) with delays is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ (i.e. $\lambda > 0$) such that

$$\left[\sum_{k=0}^{h} A_k - I_n\right] \lambda < 0. \tag{9}$$

Proof. Necessity. From (1) for $i = 0,\ 1,\ ....,p-1$ (p is a positive integer) we have respectively

$$x_1 = A_0 x_0 + \sum_{k=1}^{h} A_k x_{-k},$$

$$x_2 = A_0 x_1 + \sum_{k=1}^{h} A_k x_{1-k},$$

....

$$x_{p-1} = A_0 x_{p-2} + \sum_{k=1}^{h} A_k x_{p-2-k},$$

$$x_p = A_0 x_{p-1} + \sum_{k=1}^{h} A_k x_{p-1-k}.$$

Summing the above equalities one obtains

$$x_p + \sum_{i=1}^{p-1} x_i = A_0 \sum_{i=0}^{p-1} x_i + A_1 \sum_{i=0}^{p-1} x_{i-1} + \cdots + A_h \sum_{i=0}^{p-1} x_{i-h}.$$

The above equality can be rewritten in the form

$$x_p - A_0 x_0 = (A_0 - I_n) \sum_{i=1}^{p-1} x_i + \sum_{k=1}^{h} A_k \sum_{i=0}^{p-1} x_{i-k}.$$

Adding $-x_0$ to both sides of the above equality we obtain

$$x_p - x_0 = (A_0 - I_n) \sum_{i=0}^{p-1} x_i + \sum_{k=1}^{h} A_k \sum_{i=0}^{p-1} x_{i-k}.$$

If the system (1) is asymptotically stable then $x_p \to 0$ for $p \to \infty$. Hence

$$-x_0 = (A_0 - I_n) \sum_{i=0}^{\infty} x_i + \sum_{k=1}^{h} A_k \sum_{i=0}^{\infty} x_{i-k}$$

and

$$-x_0 - \sum_{k=1}^{h} A_k x_{-k} = \left(\sum_{k=0}^{h} A_k - I_n\right) \sum_{i=0}^{\infty} x_i. \tag{10}$$

From (2) and assumption that $x_{-k} > 0$ for at least one $k = 0, 1, ..., h$, it follows that the left hand side of (10) is strictly negative and hence

$$\left(\sum_{k=0}^{h} A_k - I_n\right) \sum_{i=0}^{\infty} x_i < 0. \tag{11}$$

The condition (11) is equivalent to (9) for $\lambda = \sum_{i=0}^{\infty} x_i > 0$.

Sufficiency. Let us consider the dual system

$$x_{i+1} = A_0^{\mathrm{T}} x_i + \sum_{k=1}^{h} A_k^{\mathrm{T}} x_{i-k}, \quad i \in Z_+, \tag{12}$$

which is positive and asymptotically stable if and only if the original system (1) is positive and asymptotically stable.

As a Lyapunov function for the dual system (12) we may choose the following function

$$V(x_i) = x_i^{\mathrm{T}} \lambda + \sum_{j=1}^{h} x_{i-j}^{\mathrm{T}} \sum_{k=j}^{h} A_k \lambda, \tag{13}$$

which is positive for non-zero $x_i \in \Re_+^n$ and for strictly positive $\lambda \in \Re_+^n$.

From (13) and (12) we have

$$\Delta V(x_i) = V(x_{i+1}) - V(x_i) =$$

$$= x_{i+1}^{\mathrm{T}} \lambda + \sum_{j=1}^{h} x_{i+1-j}^{\mathrm{T}} \sum_{k=j}^{h} A_k \lambda - x_i^{\mathrm{T}} \lambda$$

$$- \sum_{j=1}^{h} x_{i-j}^{\mathrm{T}} \sum_{k=j}^{h} A_k \lambda = x_i^{\mathrm{T}} A_0 \lambda + \sum_{j=1}^{h} x_{i+1-j}^{\mathrm{T}} \sum_{k=j}^{h} A_k \lambda$$

$$- x_i^{\mathrm{T}} \lambda - \sum_{j=1}^{h-1} x_{i-j}^{\mathrm{T}} \sum_{k=j+1}^{h} A_k \lambda = x_i^{\mathrm{T}} A_0 \lambda$$

$$+ x_i^{\mathrm{T}} \sum_{k=1}^{h} A_k \lambda - x_i^{\mathrm{T}} \lambda = x_i^{\mathrm{T}} \left[\sum_{k=0}^{h} A_k - I_n\right] \lambda < 0.$$

Hence, the condition (9) implies $\Delta V(x_i) < 0$ and the positive system (1) is asymptotically stable.

As in [20] in the case of positive systems without delays we can show that strictly positive vector $\lambda \in \Re_+^n$ may be chosen in the form $\lambda = \left[I_n - \sum_{k=0}^{h} A_n\right]^{-1} 1_n > 0$, where $1_n = [\ 1 \ \ \dots \ \ 1\ ]^{\mathrm{T}}$.

Theorem 3. The positive discrete-time system with delays (1) is asymptotically stable if and only if the positive system without delays

$$x_{i+1} = Ax_i, \quad i \in Z_+, \tag{14}$$

where

$$A = \sum_{k=0}^{h} A_k \in \Re_+^{n\times n}, \tag{15}$$

is asymptotically stable.

Proof. In [20] it was shown that the positive system (14) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ such that $[A - I_n]\lambda < 0$. Hence, the proof follows directly form the above, (15) and Theorem 2.

From Theorem 3 it follows that asymptotic stability of the positive system (1) with delays does not depend on the values and number of delays. Such a kind of stability is called as asymptotic stability independent of delay.

This means that if the positive system (1) with delays is asymptotically stable then, for example, is asymptotically stable the positive system

$$x_{i+1} = Ax_{i-p}, \quad i \in Z_+, \tag{16}$$

where A is defined by (15) and p is any positive integer or the positive system

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-d_k}, \quad i \in Z_+, \tag{17}$$

for any positive integers d_k $(k = 0, 1, ..., h)$.

From Theorem 3 and stability conditions of the positive system (14), given in [1, 2, 21], we have the following theorem and lemma.

Theorem 4. The positive system (1) with delays is asymptotically stable (independent of delays) if and only if one of the following equivalent conditions holds

1) eigenvalues $z_1, z_2, \cdots, z_n$ of the matrix A defined by (15) have moduli less than 1,
2) all leading principal minors of the matrix $I_n - A$ are positive,
3) all coefficients of the characteristic polynomial of the matrix $A - I_n$ are positive,
4) $\rho(A) < 1$, where $\rho(A) = \max_{1\le k\le n} |z_k|$ is the spectral radius of the matrix A.

Lemma 1. The positive system (1) with delays is unstable if at least one diagonal entry of the matrix A defined by (15) is greater than 1.

Now consider the positive scalar discrete-time system with delays described by the equation

$$x_{i+1} = a_0 x_i + \sum_{k=1}^{h} a_k x_{i-k}, \quad i \in Z_+, \tag{18}$$

where h is a positive integer and $a_k \ge 0, k = 0, 1, ..., h$.

From condition 2) of Theorem 4 we have directly the following result, previously obtained in [8, 11] on the basis of Theorem 1.

Lemma 2. The positive system with delays (18) is asymptotically stable if and only if $a_0 + a_1 + \cdots + a_h < 1$.

Let us consider an interval positive discrete-time system with delays, described by the equation

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k}, \quad A_k \in [A_k^-, A_k^+] \subset \Re_+^{n\times n} \quad \text{for} \quad k = 0, 1, ..., h. \tag{19}$$

The interval system is called robustly stable if the system (19) is asymptotically stable for all $A_k \in [A_k^-, A_k^+]$ $(k = 0, 1, ..., h)$.

Theorem 5. The interval positive discrete-time system (19) with delays is robustly stable if and only if at least one of the conditions of Theorem 4 holds for the matrix

$$A^+ = \sum_{k=0}^{h} A_k^+ \in \Re_+^{n\times n}. \tag{20}$$

Proof. It follows directly from the fact that the interval positive system (19) is robustly stable if and only if the positive system

$$x_{i+1} = A_0^+ x_0 + \sum_{k=1}^{h} A_k^+ x_{i-k}, \quad i \in Z_+, \tag{21}$$

is asymptotically stable [11].

From Theorem 5 and Lemma 2 we have the following simple criterion for robust stability of scalar interval positive system (see also [8, 11])

$$x_{i+1} = \sum_{k=0}^{h} a_k x_{i-k}, \quad a_k \in [a_k^-, a_k^+], \quad k = 0, 1, ..., h. \tag{22}$$

where $0 \le a_k^-$ and $a_k^- \le a_k^+$ for $k = 0, 1, ..., h$.

Lemma 3. The positive interval system (22) is robustly stable if and only if $a_0^+ + a_1^+ + \cdots + a_h^+ < 1$.

4. Illustrative examples

Example 1. Consider the positive system (1) for $n = 2, h = 1$ with the matrices

$$A_0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & a \end{bmatrix}. \tag{23}$$

Find values of the parameter $a \ge 0$ for which the system is asymptotically stable.

For the system the matrix $I_n - A = I_n - (A_0 + A_1)$ has the form

$$I_n - A = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 0.9 - a \end{bmatrix}. \tag{24}$$

Computing the leading principal minors of (24) we obtain: $\Delta_1 = 0.5 > 0$, $\Delta_2 = \det A = 0.41 - 0.5a$. Minor Δ_2 is positive if and only if $a < 0.82$. Hence, from condition 2) of Theorem 4 we have that the system is asymptotically stable if and only if $0 \leq a < 0.82$. The same result was obtained in [17] on the basis of Theorem 1.

Example 2. Find values of the parameters $a \geq 0$ and $b \geq 0$ for which is asymptotically stable the positive interval system (19) for $n = 3$, $h = 1$ with the matrices

$$A_0^- = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0^+ = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.2 & 0 & a \\ 0 & 0.1 & 0 \end{bmatrix},$$
$$A_1^- = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}, \quad A_1^+ = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.4 & 0 & 0 \\ 1 & 0 & b \end{bmatrix}. \tag{25}$$

The matrix $I_n - A^+ = I_n - (A_0^+ + A_1^+)$ has the form

$$I_n - (A_0^+ + A_1^+) = \begin{bmatrix} 1 & -0.4 & 0 \\ -0.6 & 1 & -a \\ -1 & -0.1 & 1-b \end{bmatrix}. \tag{26}$$

Computing the leading principal minors of (26) we obtain: $\Delta_1 = 1 > 0$, $\Delta_2 = 0.76 > 0$, $\Delta_3 = \det D = -0.5a + 0.76 - 0.76b$. It is easy to check that $\Delta_3 > 0$ if and only if $b < 1 - 0.5a/0.76$.

Because $a \geq 0$ and $b \geq 0$ by the assumption, from the above and condition 2) of Theorem 4 we have that the system is asymptotically stable if and only if

$$0 \leq a < 1.52 \quad \text{and} \quad 0 \leq b < 1 - 0.6579a. \tag{27}$$

The same result was obtained in [11] by applying Theorem 1 to the positive system (19) with the matrices A_0^+ and A_1^+ of the form given in (25).

5. Concluding remarks

Simple new necessary and sufficient conditions for asymptotic stability of the positive linear discrete-time systems with delays in states have been established. First, the necessary and sufficient conditions for asymptotic stability have been given in Theorems 2, 3 and 4. It has been shown that the positive system (1) with delays is asymptotically stable if and only if the positive system (14) without delays is asymptotically stable. Next, the conditions for robust stability of the positive interval systems are derived in Theorem 5. An extension of the considerations for two-dimensional linear systems with delays is an open problem.

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