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Stability of linear continuous-time fractional order systems with delays of the retarded type

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Abstract. New frequency domain methods for stability analysis of linear continuous-time fractional order systems with delays of the retarded type are given. The methods are obtained by generalisation to the class of fractional order systems with delays of the Mikhailov stability criterion and the modified Mikhailov stability criterion known from the theory of natural order systems without and with delays. The study is illustrated by numerical examples of time-delay systems of commensurate and non-commensurate fractional orders.

Key words: fractional system, linear, continuous-time, stability, delays, retarded type.

1. Introduction

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations has been the subject of many papers. For review of the previous results see [1–9], for example.

The stability problem of linear continuous-time fractional order systems without delays was studied in [5, 9–14].

Time delay systems of natural order were studied in [15–19] and of fractional order in [20–24].

The aim of the paper is to present the new frequency domain methods for stability analysis of linear continuous-time fractional order systems with delays of the retarded type. The proposed methods are based on the Mikhailov stability criterion and the modified Mikhailov stability criterion known from the stability theory of natural order systems [16, 17, 25–27].

To the best of the author knowledge, frequency domain methods for stability analysis of linear fractional order system with delays have not yet been proposed.

2. Problem formulation

Consider a linear fractional system with delays described by the transfer function

$$P(s) = \frac{q_0(s) + \sum_{j=1}^{m_2} q_j(s)\exp(-s^r\beta_j)}{p_0(s) + \sum_{i=1}^{m_1} p_i(s)\exp(-s^r h_i)} = \frac{N(s)}{D(s)}, \quad (1)$$

where r is such a real number that $0 < r \leq 1$, the fractional degree non-trivial polynomials $p_i(s)$ and $q_j(s)$ with real coefficients have the forms

$$p_i(s) = \sum_{k=0}^{n} a_{ik}s^{\alpha_k}, \quad i = 0, 1, ..., m_1, \quad (2)$$

$$q_j(s) = \sum_{k=0}^{m} b_{jk}s^{\delta_k}, \quad j = 0, 1, ..., m_2, \quad (3)$$

where α_k and δ_k are real non-negative numbers and $a_{0n} \neq 0$, $b_{0m} \neq 0$.

Without loss of generality we will assume that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$; $\delta_m > \delta_{m-1} > \cdots > \delta_1 > \delta_0 \geq 0$ and real non-negative delays β_j and h_i satisfy the inequalities $\beta_{m_2} > \beta_{m_2-1} > \cdots > \beta_1$, $h_{m_1} > h_{m_1-1} > \cdots > h_1$.

Exactly as in [21], for $s \neq 0$ and any real v, we define s^v to be $\exp(v(\log|s| + j\arg s))$, and a continuous choice of $\arg s$ in a domain leads to an analytic branch of s^v. We make normally the choice $-\pi < \arg s < \pi$ for $s \in C\backslash R_-$, where C denotes the set of complex numbers and R_- denotes the negative real axis.

The fractional degree characteristic quasi-polynomial of the system (1) has the form

$$D(s) = p_0(s) + \sum_{i=1}^{m_1} p_i(s)\exp(-s^r h_i). \quad (4)$$

By generalisation of classification of the natural degree quasi-polynomials (see [15, 17–19], for example) to the fractional degree characteristic quasi-polynomials we obtain the following.

The fractional degree characteristic quasi-polynomial (4) is

- of the retarded type if $\deg p_0(s) > \deg p_i(s)$ for all $i = 1, 2, ..., m_1$,
- of the neutral type if $\deg p_0(s) = \deg p_i(s)$ for at least one $i = 1, 2, ..., m_1$.

We will consider the time-delay systems of the retarded type, i.e. the systems satisfying the assumption $\deg p_0(s) > \deg p_i(s)$ for all $i = 1, 2, ..., m_1$.

Moreover, we assume that $\deg q_0(s) > \deg q_j(s)$, $j = 1, 2, ..., m_2$, $\deg p_0(s) > \deg q_0(s)$ in order to deal with strictly proper systems and that $N(s)$ and $D(s)$ have no common zeros.

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Theorem 1 [20]. Let $P(s)$ of the form (1) be the strictly proper transfer function, where $N(s)$ and $D(s)$ have no common zeros. Then the fractional order system described by the transfer function (1) is bounded-input bounded-output (BIBO) stable (shortly stable) if and only if $P(s)$ has no poles with non-negative real parts, i.e.

$$D(s) \neq 0 \quad \text{for} \quad \text{Re}s \geq 0. \tag{5}$$

The characteristic quasi-polynomial $D(s)$ satisfying the condition (5) we will called the stable quasi-polynomial.

Similarly as in the case of fractional order systems without delays [9], we introduce the following classification of the fractional order systems with delays.

The fractional order system with delays described by the transfer function (1) is:

- of a commensurate order if
$$\begin{aligned} \alpha_k &= k\alpha \quad (k = 0, 1, ..., n) \\ \text{and} \quad \delta_k &= k\alpha \quad (k = 0, 1, ..., m), \end{aligned} \tag{6}$$
where $\alpha > 0$ is a real number,
- of a rational order if it is of commensurate order and $\alpha = 1/q$, where q is a positive integer (in such a case $0 < \alpha \leq 1$),
- of a non-commensurate order if (6) does not hold.

A numerical algorithm for stability testing of fractional order systems with delays (of non-commensurate order, in general) was given in [23]. This algorithm is based on using the Cauchy integral theorem and solving an initial-value problem.

In this paper we give the new frequency domain necessary and sufficient conditions for stability of fractional degree characteristic quasi-polynomials (4). First, the characteristic quasi-polynomial of commensurate degree will be analysed and the frequency domain method for stability will be given. Next, the frequency domain method for stability analysis of the characteristic quasi-polynomial of non-commensurate degree will be proposed.

3. The main results

The system with delays of a fractional commensurate order is described by the transfer function (1) with

$$p_i(s) = \sum_{k=0}^{n} a_{ik} s^{k\alpha}, \quad i = 0, 1, ..., m_1, \tag{7}$$

$$q_j(s) = \sum_{k=0}^{m} b_{jk} s^{k\alpha}, \quad j = 0, 1, ..., m_2. \tag{8}$$

In such a case, applying substitution $\lambda = s^\alpha$ in (7), (8) and (1) we obtain the associated transfer function of a natural order of the form

$$\tilde{P}(\lambda) = \frac{\tilde{q}_0(\lambda) + \sum_{j=1}^{m_2} \tilde{q}_j(\lambda) \exp(-\lambda^{r/\alpha}\beta_j)}{\tilde{p}_0(\lambda) + \sum_{i=1}^{m_1} \tilde{p}_i(\lambda) \exp(-\lambda^{r/\alpha}h_i)} = \frac{\tilde{N}(\lambda)}{\tilde{D}(\lambda)}, \tag{9}$$

where

$$\tilde{p}_i(\lambda) = \sum_{k=0}^{n} a_{ik} \lambda^k, \quad i = 0, 1, ..., m_1, \tag{10a}$$

$$\tilde{q}_j(\lambda) = \sum_{k=0}^{m} b_{jk} \lambda^k, \quad j = 0, 1, ..., m_2, \tag{10b}$$

are natural number degree polynomials.

Hence, in the case of a system with delays of a fractional commensurate order we can consider the natural degree quasi-polynomial

$$\tilde{D}(\lambda) = \tilde{p}_0(\lambda) + \sum_{i=1}^{m_1} \tilde{p}_i(\lambda) \exp(-\lambda^{r/\alpha}h_i), \tag{11}$$

associated with the characteristic quasi-polynomial (4) of a fractional order.

Now we prove the following result, known in the stability theory of fractional degree polynomials [5, 9–12].

Lemma 1. The fractional quasi-polynomial (4) of commensurate degree satisfy the condition (5) if and only if all the zeros of the associated natural degree quasi-polynomial (11) satisfy the condition

$$|\arg \lambda| > \alpha \frac{\pi}{2}, \tag{12}$$

where $\arg \lambda$ denotes the main argument of the complex number λ, i.e. $\arg \lambda \in (-\pi, \pi]$.

Proof. From Theorem 1 it follows that the boundary of the stability region of fractional quasi-polynomial (4) is the imaginary axis of complex s-plane with the parametric description $s = j\omega$, $\omega \in (-\infty, \infty)$. Zeros of fractional quasi-polynomial $D(s)$ of the form (4) and the associated natural degree quasi-polynomial $\tilde{D}(\lambda)$ of the form (11) satisfy the relationship $\lambda = s^\alpha$. Hence, the boundary of the stability region in the complex λ-plane of the natural degree quasi-polynomial (11) has the parametric description

$$\lambda = (j\omega)^\alpha = |\omega|^\alpha e^{j\alpha\pi/2}, \quad \omega \in (-\infty, \infty). \tag{13}$$

All the zeros of quasi-polynomial (11) lie in the stability region with the boundary (13) if and only if (12) holds. This completes the proof.

It is easy to see that for $0 < \alpha \leq 1$ the condition (12) holds for the zeros of quasi-polynomial (11) lying in the stability region shown in Fig. 1. This region is reduced to the open left half-plane of the complex λ-plane for $\alpha = 1$.

From (12) and Fig. 1 it follows that if $1 < \alpha < 2$ then the "stability region" is a cone in the open left half-plane.

From the fundamental properties of distribution of zeros of quasi-polynomials (see, for example, [15–19]) it follows that natural degree quasi-polynomial (11) of the retarded type always has at least one chain of asymptotic zeros satisfying the conditions

$$\lim_{|\lambda|\to\infty} Re\lambda = -\infty, \quad \lim_{|\lambda|\to\infty} Im\lambda = \pm\infty.$$

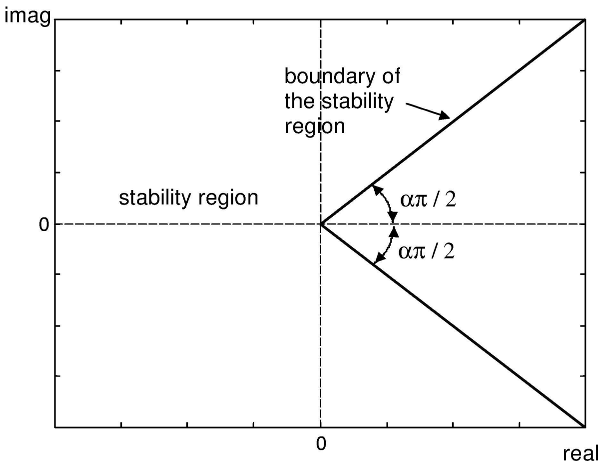

Fig. 1. Stability region of fractional quasi-polynomial (4) of commensurate degree in the complex λ-plane ($\lambda = s^{\alpha}$ with $0 < \alpha \leq 1$)

Therefore, the condition (12) with $\alpha > 1$ does not hold for the asymptotic zeros of quasi-polynomial (11). Hence, we have the following important lemma.

Lemma 2. The fractional quasi-polynomial (4) of commensurate degree (the condition (6) holds) is not stable for any $\alpha > 1$.

In the stability theory of polynomials or quasi-polynomials of natural degree, the asymptotic stability (all zeros have negative real parts) and more a general case of stability, namely the D-stability (all zeros lie in the open region D in the open left half-plane of the complex plane), are considered [16, 25].

From the above and Lemma 1 we have the following lemmas.

Lemma 3. The fractional quasi-polynomial (4) of commensurate degree is stable if and only if the associated natural degree quasi-polynomial (11) is D-stable, where parametric description of the boundary of the region D has the form (13) with $0 < \alpha \leq 1$ (see Fig. 1).

Lemma 4. The fractional quasi-polynomial (4) of commensurate degree with $0 < \alpha \leq 1$ is stable if the associated natural degree quasi-polynomial (11) is asymptotically stable, i.e. all zeros of this quasi-polynomial have negative real parts (the condition (12) holds for $\alpha = 1$).

By generalisation of the Mikhailov theorem (see [16, 25, 26], for example) to the fractional degree characteristic quasi-polynomial (4) of commensurate degree one obtains the following theorem.

Theorem 2. The fractional characteristic quasi-polynomial (4) of commensurate degree is stable if and only if

$$\underset{0 \leq \omega < \infty}{\Delta \arg} D(j\omega) = n\pi/2, \qquad (14)$$

which means that the plot of $D(j\omega)$ with ω increasing from 0 to $+\infty$ runs in the positive direction by n quadrants of the complex plane, missing the origin of this plane.

Proof. Because $\tilde{D}((j\omega)^{\alpha}) = D(j\omega)$, the condition (14) is necessary and sufficient for D-stability of natural degree quasi-polynomial (11) [16]. Hence, the proof follows from Lemma 3.

The plot of the function $D(j\omega)$, where $D(j\omega) = D(s)$ for $s = j\omega$ ($D(s)$ has the form (4)) will be called the generalised (to the class of fractional degree quasi-polynomials) Mikhailov plot.

Checking the condition of Theorem 2 is on the whole difficult, because

1) $D(j\omega)$ quickly tends to infinity as ω grows to ∞,
2) the delay terms in $D(s)$ generate an infinite number of spirals for $s = j\omega$ and $\omega \in [0, \infty)$.

Therefore, in practice Theorem 2 is not reliable. Moreover, this theorem is true only in the case of commensurate degree fractional quasi-polynomials.

To avoid difficulty 1), we introduce the rational function

$$\psi(s) = \frac{D(s)}{w_r(s)}, \qquad (15)$$

instead of fractional degree quasi-polynomial $D(s)$ of the form (4).

In (15) $w_r(s)$ is the reference fractional polynomial (or fractional quasi-polynomial) of the same degree α_n as quasi-polynomial (4). We will assume that this polynomial is stable, i.e.

$$w_r(s) \neq 0 \quad \text{for} \quad \text{Re}s \geq 0. \qquad (16)$$

The reference fractional polynomial $w_r(s)$ can be chosen in the form

$$w_r(s) = a_{0n}(s + c)^{\alpha_n}, \quad c > 0, \qquad (17)$$

where a_{0n} is the coefficient of s^{α_n} in polynomial $p_0(s)$ of the form (2) for $i = 0$.

Note that the reference polynomial (17) is stable for $c > 0$,

The main result of the paper is as follows.

Theorem 3. The fractional characteristic quasi-polynomial (4) (of commensurate or non-commensurate degree) is stable if and only if

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} \psi(j\omega) = 0. \qquad (18)$$

Proof. From (15) for $s = j\omega$ it follows that

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} \psi(j\omega) = \underset{\omega \in (-\infty, \infty)}{\Delta \arg} D(j\omega) - \underset{\omega \in (-\infty, \infty)}{\Delta \arg} w_r(j\omega). \qquad (19)$$

The reference polynomial $w_r(s)$ of the same fractional degree as quasi-polynomial (4) is stable by assumption. Therefore, the fractional quasi-polynomial (4) is stable if and only if

$$\underset{\omega \in (-\infty, \infty)}{\Delta \arg} D(j\omega) = \underset{\omega \in (-\infty, \infty)}{\Delta \arg} w_r(j\omega), \qquad (20)$$

which holds if and only if (18) is satisfied.

The plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$ ($\psi(s)$ is defined by (15)) we will call the generalised modified Mikhailov plot.

The condition (18) of Theorem 3 holds if and only if the generalised modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane as ω runs from $-\infty$ to ∞.

Form (15), (4) and (17) we have

$$\lim_{\omega\to\pm\infty}\psi(j\omega)=\lim_{\omega\to\pm\infty}\frac{D(j\omega)}{w_r(j\omega)}=1 \tag{21a}$$

and

$$\psi(0)=\frac{D(0)}{w_r(0)}=\frac{a_{00}+a_{10}+\cdots+a_{m_10}}{a_{0n}c^{\alpha_n}}. \tag{21b}$$

From the above it follows that the generalised modified Mikhailov plot encircles or crosses the origin of the complex plane if $\psi(0)\le 0$. Hence, we have the following lemma.

Lemma 5. The fractional characteristic quasi-polynomial (4) of commensurate or non-commensurate degree is not stable if

$$\frac{a_{00}+a_{10}+\cdots+a_{m_10}}{a_{0n}}\le 0.$$

Theorem 3 and Lemma 5 are the generalisation to the fractional quasi-polynomials case of results given in [10] and [11] for the fractional polynomials (of commensurate and non-commensurate degree, respectively).

4. Illustrative examples

Example 1. Check the stability of the fractional order system with delays and with characteristic quasi-polynomial of the form

$$D(s)=s^{3/2}-1.5s-1.5s\exp(-sh)+4s^{1/2}+8. \tag{22}$$

For $\alpha=1/2$ and $\lambda=s^\alpha=s^{1/2}$ from (22) one obtains the following associated fractional quasi-polynomial of natural degree

$$\tilde{D}(\lambda)=\lambda^3-1.5\lambda^2-1.5\lambda^2\exp(-\lambda^2h)+4\lambda+8. \tag{23}$$

From Lemma 3 it follows that fractional quasi-polynomial (22) of commensurate degree is stable (the condition (5) holds) if and only if the natural degree quasi-polynomial (23) is D-stable, where the region D is shown in Figure 1 with $\alpha=1/2$.

Substituting $h=0$ in (22) and (23) we obtain, respectively, the fractional and natural degree polynomials $D(s)=s^{3/2}-3s+4s^{1/2}+8$ and $\tilde{D}(\lambda)=\lambda^3-3\lambda^2+4\lambda+8$. Polynomial $\tilde{D}(\lambda)$ has the following zeros: $\lambda_1=-1$, $\lambda_{2,3}=2\pm j2$. Zeros λ_2 and λ_3 lie on the boundary of D-stability region, which means that polynomial $\tilde{D}(\lambda)$ is not D-stable and the fractional quasi-polynomial (22) is not stable for $h=0$. In [24] it was shown (see also [23]) that the fractional quasi-polynomial (22) is stable for a few intervals of values of the delay h, where $H_1=(0.04986,\ 0.78539)$ is the first interval of stability.

We check the stability of the fractional characteristic quasi-polynomial (22) with $h=0.1$.

The plot of the function (15) for $s=j\omega$ and $\omega\in[0,\ 500]$, where $D(s)$ has the form (22) for $h=0.1$ and $w_r(s)=(s+5)^{3/2}$, is shown in Figure 2. According to (21) we have $\psi(0)=8/5^{3/2}=0.7155$, $\lim_{\omega\to\infty}\psi(j\omega)=1$. The plot is symmetrical with respect to the real axis for negative values of frequency ω. This plot does not encircle the origin of the complex plane, which means that the fractional system with characteristic quasi-polynomial (22) with $h=0.1$ is stable, according to Theorem 3.

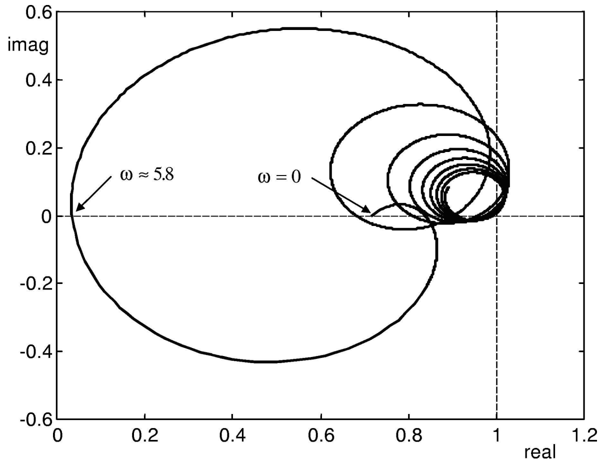

Fig. 2. Plot of (15) for $s=j\omega$, $\omega\in[0,\ 500)$, $w_r(s)=(s+5)^{3/2}$

Example 2. Consider the control system shown in Figure 3 with a fractional order plant described by the transfer function

$$P(s)=\frac{e^{-0.5s}}{1+s^{0.5}} \tag{24}$$

and fractional PID controller

$$C(s)=K+\frac{I}{s^\lambda}+Ds^\mu, \tag{25}$$

where $\lambda=1.1011$, $\mu=0.1855$, $K=1.4098$, $I=1.6486$, $D=-0.2139$ [8].

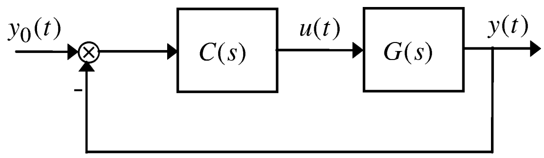

Fig. 3. The feedback control system

The characteristic quasi-polynomial of the closed-loop system has the form

$$\begin{aligned}D(s)&=s^{\lambda+1/2}+s^\lambda+(Ks^\lambda+Ds^{\lambda+\mu}+I)e^{-0.5s}\\&=s^{1.6011}+s^{1.1011}+(1.4098s^{1.1011}-0.2139s^{1.2866}\\&\quad+1.6486)e^{-0.5s}.\end{aligned} \tag{26}$$

The control system with characteristic quasi-polynomial (26) is stable if and only if all zeros of (26) have negative real parts.

For stability analysis we apply Theorem 3 with the reference polynomial $w_r(s)=(s+10)^{1.6011}$. In this case the function (15) has the form

$$\psi(s)=\frac{D(s)}{(s+10)^{1.6011}}, \tag{27}$$

where $D(s)$ is given by (26).

The plot of function (27) for $s=j\omega$, $\omega\in(-\infty,\ \infty)$, is shown in Fig. 4, where

$$\lim_{\omega\to\pm\infty}\psi(j\omega)=1,\quad \psi(0)=1.6486\ /\ 10^{1.6011}=0.0413. \tag{28}$$

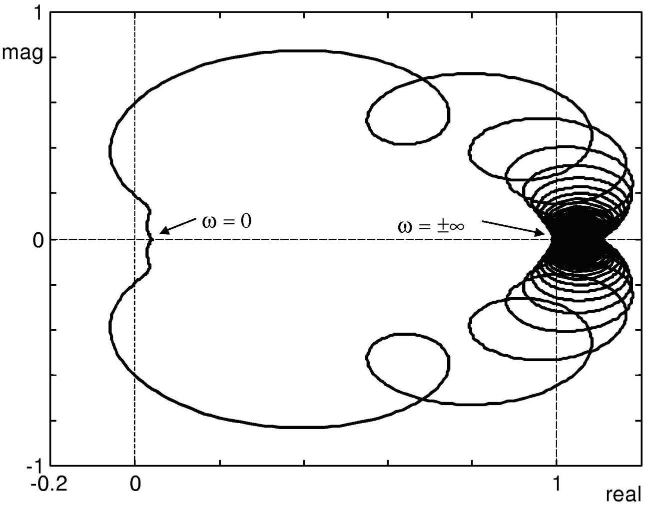

Fig. 4. Plot of the function (27) for $s = j\omega$, $\omega \in (-\infty, \infty)$

From Fig. 4 it follows that the generalised modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane. This means that the fractional control system with characteristic quasi-polynomial (26) is stable, according to Theorem 3.

5. Concluding remarks

New frequency domain methods for stability analysis of linear fractional order systems with delays of the retarded type have been given. The methods have been obtained by generalisation of the Mikhailov stability criterion and the modified Mikhailov stability criterion (known from the theory of natural order systems without and with delays) for the case of fractional order systems with delays.

In particular it has been shown that:

- the fractional quasi-polynomial of commensurate degree (of the form (4) with (7)) is unstable for $\alpha > 1$ (Lemma 2),
- the fractional quasi-polynomial of commensurate degree is stable if and only if the associated natural degree quasi-polynomial (11) is D-stable, where the stability region D is shown in Figure 1 with $0 < \alpha \leq 1$ (Lemma 3),
- the fractional characteristic polynomial (4) (of commensurate or non-commensurate degree) is stable if and only if the plot of the rational function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$, where $\psi(s)$ is defined by (15), called the generalised modified Mikhailov plot, does not encircle the origin of the complex plane (Theorem 3).

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