

# Positive fractional 2D hybrid linear systems

T. KACZOREK\*

Faculty of Electrical Engineering, Białystok Technical University, 45D Wiejska St, 15-351 Białystok, Poland

**Abstract.** A new class of positive fractional 2D hybrid linear systems is introduced. The solution of the hybrid system is derived. The classical Cayley-Hamilton theorem is extended for fractional 2D hybrid systems. Necessary and sufficient conditions for the positivity are established.

**Key words:** positive, partial realization, existence, computation, linear, discrete-time, system.

## 1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. Recent developments in positive discrete-time and continuous-time systems without and with delays was considered in [1, 2–9].

The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [10].

The relative controllability of stationary hybrid systems has been investigated in [11] and the observability of linear differential-algebraic systems with delays has been considered in [12].

The positive 2D hybrid linear systems have been investigated in [13].

The main purpose of this paper is to introduce a class of fractional 2D hybrid systems. A solution to the hybrid system will be derived. The classical Cayley-Hamilton theorem will be extended for fractional hybrid systems. Necessary and sufficient conditions for the positivity will be established.

To the best knowledge of the author the positive fractional 2D hybrid linear systems have not been considered yet.

## 2. Equations of the fractional 2D hybrid systems and their solutions

Let  $\mathfrak{R}^{n \times m}$  be the set of  $n \times m$  real matrices with entries from the real number  $\mathfrak{R}$  and  $Z_+$  be the set of nonnegative integers. The  $n \times m$  identity matrix will be denoted by  $I_n$ .

Consider a hybrid fractional 2D system described by the equations

$$\frac{d^\alpha x_1(t, i)}{dt^\alpha} = A_{11}x_1(t, i) + A_{12}x_2(t, i) + B_1u(t, i), \quad (1a)$$

$$t \in \mathfrak{R}_+ = [0, +\infty]$$

$$\Delta^\beta x_2(t, i + 1) = A_{21}x_1(t, i) + A_{22}x_2(t, i) + B_2u(t, i), \quad (1b)$$

$$i \in Z_+$$

$$y(t, i) = C_1x_1(t, i) + C_2x_2(t, i) + Du(t, i) \quad (1c)$$

where  $\alpha$  ( $0 < \alpha < 1$ ) is the order of fractional derivative,  $\beta$  ( $0 < \beta < 1$ ) is the order of fractional difference,  $x_1(t, i) \in \mathfrak{R}^{n_1}$ ,  $x_2(t, i) \in \mathfrak{R}^{n_2}$ ,  $u(t, i) \in \mathfrak{R}^m$ ,  $y(t, i) \in \mathfrak{R}^p$  and  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$ ,  $B_1$ ,  $B_2$ ,  $C_1$ ,  $C_2$ ,  $D$  are real matrices with appropriate dimensions.

Boundary conditions for (1a) and (1b) have the form

$$x_1(0, i) = x_1(i), \quad (2)$$

$$i \in Z_+ \quad \text{and} \quad x_2(t, 0) = x_2(i), \quad t \in \mathfrak{R}_+$$

Note that fractional 2D hybrid system (1) has a similar structure as the Roesser model [2, 13, 14].

The Caputo definition of the fractional derivative [14]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau$$

$$\left( x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n} \right) \quad n - 1 < \alpha < n \in N = \{1, 2, \dots\} \quad (3)$$

\*e-mail: kaczonek@isep.pw.edu.pl

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{4}$$

is the gamma function will be used.

The fractional difference of the order  $\beta$  of  $x_i$  will be defined by

$$\Delta^\beta x_i = \sum_{k=0}^i (-1)^k \binom{\beta}{k} x_{i-k}, \tag{5}$$

$$0 < \beta < 1, i \in Z_+$$

where

$$\binom{\beta}{k} = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\beta(\beta-1)\dots(\beta-k+1)}{k!} & \text{for } k = 1, 2, \dots \end{cases} \tag{6}$$

Using (5) we may write the equation (1b) in the form

$$x_2(t, i+1) = A_{21}x_1(t, i) + A_{22}x_2(t, i) + \sum_{k=1}^{i+1} c_k x_2(t, i-k+1) + B_2 u(t, i), \quad i \in Z_+ \tag{7}$$

where

$$c_k = c_k(\beta) = (-1)^{k-1} \binom{\beta}{k}, \quad k = 1, 2, \dots \tag{8}$$

**Remark.** From (6) and (8) it follows that coefficients  $c_k$  strongly decrease when  $k$  increases. Therefore, in practical problems it is assumed that  $i$  is bounded by a natural number

$$L \text{ and } \sum_{k=1}^{i+1} c_k x_2(t, i-k+1) = \sum_{k=1}^{L+1} c_k x_2(t, i-k+1).$$

**Theorem 1.** Solutions to the equations (1a) and (7) with given boundary conditions (2) have the forms

$$\begin{bmatrix} x_1(t, i) \\ x_2(t, i) \end{bmatrix} = \sum_{p=0}^\infty \sum_{q=0}^i \frac{T_{p,i-q} B_{10}}{\Gamma[(p+1)\alpha]} \int_0^t (t-\tau)^{(p+1)\alpha-1} u(\tau, q) d\tau + \sum_{p=0}^\infty \sum_{q=0}^i \frac{T_{p,i-q-1} B_{01}}{\Gamma(\alpha p - 1)} \int_0^t (t-\tau)^{\alpha p - 1} u(\tau, q) d\tau + \sum_{p=0}^\infty \sum_{q=0}^i \frac{T_{p,i-q}}{\Gamma(\alpha p + 1)} (t-\tau)^{p\alpha} \begin{bmatrix} x_1(0, q) \\ 0 \end{bmatrix} + \sum_{p=0}^\infty \frac{T_{p,i}}{\Gamma(\alpha p)} \int_0^t (t-\tau)^{p\alpha} \begin{bmatrix} 0 \\ x_2(t, 0) \end{bmatrix} d\tau \tag{9}$$

where

$$T_{pq} = \begin{cases} I_{n_1+n_2} & \text{for } p = q = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } p = 1, q = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2} c_1 \end{bmatrix} & \text{for } p = 0, q = 1 \\ T_{10} T_{p-1,q} + T_{01} T_{p,q-1} & \text{for } p \geq 0, q \geq 0, p+q > 1 \\ 0 & \text{for } p < 0, \text{ or/and } q < 0 \end{cases} \tag{10a}$$

and

$$T_{0q} = T_{01}^q + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} c_q \end{bmatrix}, \quad q = 2, 3, \dots \tag{10b}$$

**Proof.** The solutions will be derived using the Laplace transform ( $\mathcal{L}$ ) with respect to  $t$  and the  $Z$  transform ( $\mathcal{Z}$ ) with respect to  $i$ .

Taking into account that

$$\begin{aligned} \mathcal{L} \left[ \frac{d^\alpha x_1(t, i)}{dt^\alpha} \right] &= s^\alpha X_1(s, i) - s^{\alpha-1} x_1(0, i) \\ \mathcal{Z} \mathcal{L} \left[ \frac{d^\alpha x_1(t, i)}{dt^\alpha} \right] &= Z [s^\alpha X_1(s, i) - s^{\alpha-1} x_1(0, i)] \\ &= s^\alpha X_1(s, z) - s^{\alpha-1} X_1(0, z) \\ \mathcal{Z} \mathcal{L} [x_2(t, i - k + 1)] &= z^{1-k} X_2(s, z) \quad (k \geq 1) \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathcal{L} [x(t, i)] &= \int_0^\infty x(t, i) e^{-st} dt, \\ \mathcal{Z} [x(t, i)] &= \sum_{i=0}^\infty x(t, i) z^{-i} \end{aligned} \tag{12}$$

from (1a) and (7) we obtain

$$\begin{aligned} s^\alpha X_1(s, z) - s^{\alpha-1} X_1(0, z) &= A_{11} X_1(s, z) \\ &+ A_{12} X_2(s, z) + B_1 U(s, z) \\ z X_2(s, z) - z X_2(s, 0) &= A_{21} X_1(s, z) + A_{22} X_2(s, z) \\ &+ \sum_{k=1}^{i+1} c_k z^{1-k} X_2(s, z) + B_2 U(s, z) \end{aligned} \tag{13}$$

where  $U(s, z) = \mathcal{Z} \mathcal{L} [u(i, t)]$ .

From (13) we have

$$\begin{aligned} \begin{bmatrix} X_1(s, z) \\ X_2(s, z) \end{bmatrix} &= \\ &= \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1} \\ &\quad \left\{ \begin{bmatrix} B_1s^{-\alpha} \\ B_2z^{-1} \end{bmatrix} U(s, z) + \begin{bmatrix} s^{-1}X_1(0, z) \\ X_2(s, 0) \end{bmatrix} \right\}. \end{aligned} \tag{14}$$

Let

$$\begin{aligned} \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1} &= \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} s^{-\alpha p} z^{-q}. \end{aligned} \tag{15}$$

From definition of inverse matrix and (15) we have

$$\begin{aligned} \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1} &= \\ \left( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} s^{-\alpha p} z^{-q} \right) &= \left( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} s^{-\alpha p} z^{-q} \right) \\ \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1} &= \\ &= I_{n_1+n_2}. \end{aligned} \tag{16}$$

Comparison of the coefficients at the same powers of  $s$  and  $z$  of the equality (16) yields (10).

Substituting (15) into (14) we obtain

$$\begin{aligned} \begin{bmatrix} X_1(s, z) \\ X_2(s, z) \end{bmatrix} &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} s^{-\alpha p} z^{-q} \\ \left\{ \begin{bmatrix} B_1s^{-\alpha} \\ B_2z^{-1} \end{bmatrix} U(s, z) + \begin{bmatrix} s^{-1}X_1(0, z) \\ X_2(s, 0) \end{bmatrix} \right\} &= \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left( T_{pq} s^{-\alpha(p+1)} z^{-q} B_{10} + T_{pq} s^{-\alpha p} z^{-(q+1)} B_{01} \right) \\ &\quad U(s, z) + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left( T_{pq} s^{-(\alpha p+1)} z^{-q} \begin{bmatrix} X_1(0, z) \\ 0 \end{bmatrix} \right. \\ &\quad \left. + T_{pq} s^{-\alpha p} z^{-q} \begin{bmatrix} 0 \\ X_2(s, 0) \end{bmatrix} \right) \end{aligned} \tag{17}$$

where

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ \mathbf{B}_2 \end{bmatrix}.$$

Applying the inverse transforms to (17) and taking into account that

$$\mathcal{L}[t^\alpha] = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}} \tag{18}$$

we obtain (9).  $\square$

### 3. Extension of the Cayley-Hamilton theorem for the fractional 2D hybrid systems

Taking into account Remark we may write

$$\begin{aligned} \det \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix} &= \\ &= \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{N_1-k, N_2-l} s^{-\alpha k} z^{-l} \end{aligned} \tag{19}$$

for the same natural numbers  $N_1, N_2$ .

**Theorem 2.** Let (19) be the characteristic polynomial of the fractional 2D hybrid system. Then the matrices  $T_{pq}$  defined by (10) satisfy the equation

$$\sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{kl} T_{kl} = 0. \tag{20}$$

**Proof.** From the definition of inverse matrix and (19), (15) we have

$$\begin{aligned} \text{Adj} \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix} &= \\ &= \left( \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{N_1-k, N_2-l} s^{-\alpha k} z^{-l} \right) \left( \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{pq} s^{-\alpha p} z^{-q} \right) \end{aligned} \tag{21}$$

where  $\text{Adj } M$  denotes the adjoint matrix of  $M$ .

Comparison of the coefficients at the same powers  $s^{-\alpha N_1} z^{-N_2}$  of the equality (21) yields (20) since the degrees of the left-hand side are less than  $\alpha N_1$  and  $N_2$ .

Theorem 2 is an extension of the well-known classical Cayley-Hamilton theorem for the fractional 2D hybrid systems.

### 4. Positive fractional 2D hybrid systems

Let  $\mathfrak{R}_+^{n \times m}$  be the set of  $n \times m$  real matrices with nonnegative entries  $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$ .

**Definition 1.** The fractional 2D hybrid system (1) is called (internally) positive if and only if  $x_1(t, i) \in \mathfrak{R}_+^{n_1}, x_2(t, i) \in \mathfrak{R}_+^{n_2}, t \geq 0, i \in Z_+$  for any boundary conditions

$$x_1(0, i) = x_1(i) \in \mathfrak{R}_+^{n_1}, \tag{22}$$

$$i \in Z_+ \text{ and } x_2(t, 0) = x_2(i) \in \mathfrak{R}_+^{n_2}, \quad t \geq 0$$

and all inputs  $u(t, i) \in \mathfrak{R}_+^m, t \geq 0, i \in Z_+$ .

**Lemma 1.** If  $0 < \beta < 1$  then

$$c_k = c_k(\beta) > 0 \text{ for } k = 1, 2, \dots \tag{23}$$

**Proof.** The hypothesis is true for  $k = 1$  since from (8) and (6) we have  $c_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix} = \beta$ . Assuming that the hypothesis is true for  $k \geq 1$  we shall show that it is also true for  $k + 1$ . From (8) and (6) we obtain

$$\begin{aligned} c_{k+1} &= (-1)^k \begin{pmatrix} \beta \\ k+1 \end{pmatrix} \\ &= (-1)^{k-1} \frac{\beta(\beta-1)\dots(\beta-k+1)(k-\beta)}{k!(k+1)} \\ &= (-1)^{k-1} \begin{pmatrix} \beta \\ k \end{pmatrix} \frac{k-\beta}{k+1} = c_k \frac{k-\beta}{k+1} > 0 \end{aligned}$$

since  $c_k > 0$  for  $k \geq 1$ .

**Lemma 2.** If  $0 < \beta < 1$  and  $A_{21} \in \mathfrak{R}_+^{n_2 \times n_1}, A_{22} \in \mathfrak{R}_+^{n_2 \times n_2}$  then

$$T_{0q} \in \mathfrak{R}_+^{(n_1+n_2) \times (n_1+n_2)} \text{ for } q = 2, 3, \dots \tag{24}$$

**Proof.** From (10b) and Lemma 1 we have

$$\begin{aligned} T_{0q} &= \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2} c_1 \end{bmatrix}^q \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} c_q \end{bmatrix} \in \mathfrak{R}_+^{(n_1+n_2) \times (n_1+n_2)} \text{ for } q = 2, 3, \dots \end{aligned}$$

since  $c_k > 0$  for  $k = 1, 2, \dots$

Let  $M_n$  be the set of  $n \times n$  Metzler matrices (real matrices with nonnegative off-diagonal entries).

**Theorem 3.** The fractional 2D hybrid system (1) is (internally) positive if and only if

$$\begin{aligned} A_{11} &\in M_{n_1}, \quad A_{12} \in \mathfrak{R}_+^{n_1 \times n_2}, \quad A_{21} \in \mathfrak{R}_+^{n_2 \times n_1}, \\ A_{22} &\in \mathfrak{R}_+^{n_2 \times n_2}, \quad B_1 \in \mathfrak{R}_+^{n_1 \times m}, \quad B_2 \in \mathfrak{R}_+^{n_2 \times m}, \\ C_1 &\in \mathfrak{R}_+^{p \times n_1}, \quad C_2 \in \mathfrak{R}_+^{p \times n_2}, \quad D \in \mathfrak{R}_+^{p \times m}. \end{aligned} \tag{25}$$

**Proof.** From (1a) for and (22) we have [14]

$$\frac{d^\alpha x_1(t, 0)}{dt^\alpha} = A_{11}x_1(t, 0) + A_{12}x_2(t) + B_1u(t, 0) \tag{26}$$

and

$$\begin{aligned} x_1(t, 0) &= \Phi_0(t)x_1(0) \\ &+ \int_0^t \Phi(t-\tau)[A_{12}x_2(\tau) + B_1u(\tau, 0)]d\tau \end{aligned} \tag{27}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_{11}^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{28a}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A_{11}^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{28b}$$

In [14] was shown that  $\Phi_0(t) \in \mathfrak{R}_+^{n_1 \times n_1}$  and  $\Phi(t) \in \mathfrak{R}_+^{n_1 \times n_1}$  for  $t \geq 0$  if and only if  $A_{11}$  is a Metzler matrix.

From (7) for  $i = 0$  we obtain

$$x_2(t, 1) = A_{21}x_1(t, 0) + (A_{22} + I_{n_2}c_1)x_2(t) + B_2u(t, 0) \tag{29}$$

and after substitution of (27) into (29)

$$\begin{aligned} x_2(t, 1) &= A_{21}\Phi_0(t)x_1(0) + A_{21} \int_0^t \Phi(t-\tau) \\ &[A_{12}x_2(\tau) + B_1u(\tau, 0)]d\tau \\ &+ (A_{22} + I_{n_2}c_1)x_2(t) + B_2u(t, 0). \end{aligned} \tag{30}$$

From (30) for (22) it follows that  $x_2(t, 1) \in \mathfrak{R}_+^{n_2}$  if and only if  $A_{11} \in M_{n_1}, A_{21} \in \mathfrak{R}_+^{n_2 \times n_1}, A_{22} \in \mathfrak{R}_+^{n_2 \times n_2}, B_1 \in \mathfrak{R}_+^{n_1 \times m}, B_2 \in \mathfrak{R}_+^{n_2 \times m}$  and  $u(t, 0) \in \mathfrak{R}_+^m, t \geq 0$ .

Likewise from (1a) for we obtain

$$\frac{d^\alpha x_1(t, 1)}{dt^\alpha} = A_{11}x_1(t, 1) + A_{12}x_2(t, 1) + B_1u(t, 1) \tag{31}$$

and

$$\begin{aligned} x_1(t, 1) &= \Phi_0(t)x_1(0, 1) \\ &+ \int_0^t \Phi(t-\tau)[A_{12}x_2(\tau, 1) + B_1u(\tau, 1)]d\tau. \end{aligned} \tag{32}$$

From (32) it follows that  $x_1(t, 1) \in \mathfrak{R}_+^{n_1}, t \geq 0$  if and only if  $A_{11} \in M_{n_1}$  since  $x_1(0, 1) \in \mathfrak{R}_+^{n_1}$  and  $x_2(t, 1) \in \mathfrak{R}_+^{n_2}, t \geq 0$ .

Continuing this procedure it can be shown that the fractional 2D hybrid system is positive if and only if the conditions (25) are met.

### 5. Concluding remarks

A new class of fractional 2D hybrid positive linear systems has been introduced. Necessary and sufficient conditions for the positivity of the hybrid 2D linear systems has been established. The classical Cayley-Hamilton theorem has been extended for the hybrid systems. Following [14] the sufficient conditions for the reachability and controllability [15, 16] can be extended for the fractional 2D positive hybrid systems.

An open problem is extension of the considerations for 2D hybrid systems described by models with structure similar to the Kurek model [2, 17, 18].

**Acknowledgements.** The work was supported by the State Committee for Scientific Research of Poland under grant No. N N514 1939 33.

REFERENCES

[1] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, J. Wiley, New York, 2000.

[2] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[3] T. Kaczorek, "Some recent developments in positive systems", *Proc. 7<sup>th</sup> Conf. Dynamical Systems Theory and Applications*, 25–35 (2003).

[4] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem", *IEEE Trans. Autom. Control* 49 (5), 651–664, 2004.

[5] T. Kaczorek, "A realization problem for positive continuous-time linear systems with reduced number of delay", *Int. J. Appl. Math. Comp. Sci.* 3, 325–331, 2006.

[6] T. Kaczorek, "Realization problem for positive continuous-time systems with reduced numbers of delays", *Int. J. Appl. Math. Comput. Sci.* 16 (3), 101–107 (2006).

[7] T. Kaczorek, "Realization problem for positive discrete-time systems with delay", *System Science* 30 (4), 17–30 (2004).

[8] T. Kaczorek, *Positive minimal realizations for singular discrete-time systems with delays in state and control*, Bull. Pol. Ac.: Tech., 53 (3), 293–298 (2005).

[9] T. Kaczorek and M. Busłowicz, "Minimal realization problem for positive multivariable linear systems with delay", *Int. J. Appl. Math. Comput. Sci.* 14 (2), 181–187 (2004).

[10] T. Kaczorek and M. Busłowicz, "Reachability and minimum energy control of positive linear discrete-time systems with multiple delays in state and controls", *Measurements Automatics Control* 53 (10), 1–5 (2007), (in Polish).

[11] V.M. Marchenko and O.N. Poddubnaya, "Relative controllability of stationary hybrid systems", *10<sup>th</sup> IEEE Int. Conf. Methods and Models in Automation and Robotics*, 267–272 (2004).

[12] V.M. Marchenko, O.N. Poddubnaya, and Z. Zaczekiewicz, "On the observability of linear differential-algebraic systems with delays", *IEEE Trans. Autom. Contr.* 51 (8), 1387–1392 (2006).

[13] T. Kaczorek, "Positive 2D hybrid linear systems", *Bull. Pol. Ac.: Tech.* 55 (4), 351–358 (2007).

[14] T. Kaczorek, "Fractional positive continuous-time linear systems and their reachability", *Int. J. Appl. Math. Sci.* 1, (2008).

[15] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic Publ., Dordrecht, 1991.

[16] R.B. Roesser, "A discrete state-space model for linear image processing", *IEEE Trans. Autom. Contr.* AC-20 (1), 1–10 (1975).

[17] J. Kurek, "The general state-space model for a two-dimensional linear digital system", *IEEE Trans. Autom. Contr.* AC-30, 600–602 (1985).

[18] M.E. Valcher, "On the initial stability and asymptotic behaviour of 2D positive systems", *IEEE Trans on Circuits and Systems* 44 (7), 602–613 (1997).