BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 56, No. 1, 2008

The article is dedicated to
Professor Henryk Górecki
on the celebration of His 80th birthday

Strong stability of positive and compartmental linear systems

T. KACZOREK*

Institute of Control and Industrial Electronics, Warsaw University of Technology,
75 Koszykowa St., 00-662 Warsaw, Poland

Abstract. The concept of strong stability is extended for positive and compartmental linear systems. It is shown that: 1) the asymptotically stable positive and compartmental systems are strongly stable if the eigenvalues of the system matrix are distinct, 2) electrical circuits consisting of resistances, capacitances (inductances) and source voltages are strongly stable.

Key words: strong stability, positive, compartmental, linear, system, electrical circuit.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems in more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. The compartmental systems are a special subsets of the positive systems [3–4].

The concept of strong stability of linear time-invariant systems are introduced in the paper [5]. The strong stability has been related to the asymptotic stability, system structure and skewness of eigenframe and the state-space transformations under which the strong stability is a system invariant has been also characterized.

In this note the concept of strong stability will be extended for positive and compartmental linear systems. It will be shown that the asymptotically stable positive and compartmental systems are strongly stable if the eigenvalues of the system matrix are distinct and electrical circuits consisting of resistances, capacitances (inductances) and source voltages are strongly stable.

To the best knowledge of the author the strong stability of positive and compartmental linear systems has not been considered yet.

2. Positive linear systems

In this section we recall the basic definitions and theorems concerning the positive linear systems.

Let $R^{n\times m}$ be the set of real $n\times m$ matrices with and $R^n = R^{n\times 1}$. The set of real $n\times m$ matrices with nonnegative entries will be denoted by $R_+^{n\times m}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix will be denoted by I_n.

Consider the linear continuous-time system:

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where, $x = x(t) \in R^n$, $u = u(t) \in R^m$, $y = y(t) \in R^p$ are the state, input and output vectors and, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.
The system (1) is called (internally) positive if $x(t) \in R_+^n$ and $y(t) \in R_+^p$, $t \geq 0$ for any $x_0 \in R_+^n$ and every $u(t) \in R_+^m$, $t \geq 0$.
$A = [a_{ij}] \in R_+^{n\times n}$ is called the Metzler matrix if its off-diagonal entries are nonnegative, $a_{ij} \geq 0$ for $i \neq j$, $i, j = 1, ..., n$.

Theorem 1 [1, 4]. The system (1) is positive if and only if A is a Metzler matrix and $B \in R_+^{n\times m}$, $C \in R_+^{p\times n}$, $D \in R_+^{p\times m}$.

The positive continuous-time system (1) is asymptotically stable if and only if all eigenvalues of the Metzler matrix A have negative real parts [1, 2].
Consider the linear discrete-time system:

$$x_{i+1} = Ax_i + Bb_i \qquad i \in Z_+, \tag{2a}$$

$$y_i = Cx_i + Du_i \tag{2b}$$

where,$x_i \in R^n$, $u_i \in R^m$, $y_i \in R^p$ are the state, input and output vectors and, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

The system (2) is called (internally) positive if $x_i \in R_+^n$, $y_i \in R_+^p$, $i \in Z_+$ for any $x_0 \in R_+^n$ and every $u_i \in R_+^m$, $i \in Z_+$.

Theorem 2 [1, 4]. The system (2) is positive if and only if

$$A \in R_+^{n\times n}, \quad B \in R_+^{n\times m}, \quad C \in R_+^{p\times n}, \quad D \in R_+^{p\times m} \tag{3}$$

*e-mail: kaczorek@isep.pw.edu.pl

3. Strong stability

Consider the positive autonomous continuous-time system

$$\dot{x} = Ax \tag{4}$$

where $A \in R^{nxn}$ is a Metzler matrix.

Let $\|x\|$ be the Euclidean norm of the state vector $x = x(t)$ and $S_0, r)$ be the sphere with the radius $r = \|x\|^{\frac{1}{2}}$ and origin at $x = 0$.

The system (4) exhibits state-space overshoots it for at least one initial condition x_0 in the sphere $S(0,r)$ the trajectory of the system $x(t)$ satisfies the condition $\|x(t)\| > r$ for some interval $[t_0, t_1]$ [5].

In [5] it was shown that the system (4) has no overshoot for initial conditions in the sphere $S(0,r)$ if the quadratic form x^TAx is negative definite. It is well-known that $x^TAx = x^T\bar{A}x$ where $\bar{A} = \frac{1}{2}(A + A^T)$. In [5] the conditions were given such that $\bar{A}$ is negative definite.

Definition 1. The system (4) is called strongly stable if for any initial condition $x_0 \in R_+^n$ in the sphere $S(0,r)$ the following conditions are satisfied:

i. $\rho(x_0, t) \le \rho(x_0, 0)$ for $t \ge 0$ and $x_0 \in S(0,r)$

ii. $\lim\limits_{t\to\infty} \rho(x_0, t) = 0$

If the positive system (4) is strongly stable then the system has no overshoot for any initial conditions.

Theorem 3. Let the positive system (4) be asymptotically stable. Then the system is strongly stable if the matrix has distinct eigenvalues.

Proof. If the matrix A has distinct eigenvalues, $\lambda_k \neq \lambda_j$ for $k \neq j$, $\lambda_k = -\alpha_k + i\beta_k$, $k = 1, ..., n$, $i = \sqrt{-1}$ then from the Sylvester formula [6, 7] we have

$$e^{At} = \sum_{k=1}^{n} Z_k e^{\lambda_k t} \tag{5}$$

where

$$Z_k = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{A - I_n\lambda_j}{\lambda_k - \lambda_j}, \quad k = 1, ..., n \tag{6}$$

$$\sum_{k=1}^{n} Z_k = I_n \tag{7}$$

From (5) and (7) we have

$$x(t) = e^{At}x_0 = \sum_{k=1}^{n} Z_k e^{\lambda_k t} x_0 \le \left(\sum_{k=1}^{n} Z_k\right) x_0 e^{-\alpha t} = x_0 e^{-\alpha t}, \quad t \ge 0 \tag{8}$$

where

$$\alpha = \min_{1\le k\le n} \alpha_k \tag{9}$$

By assumption the positive system (4) is asymptotically stable ($\alpha < 0$) and from (8) we obtain

$$\rho(x_0, t) \le \rho(x_0, 0)e^{-\alpha t} \tag{10}$$

and this implies ii.

The following example shows that if the matrix has at least one multiple eigenvalue then the system may not be strongly stable.

4. Example

Consider the positive system (4) with the Metzler matrix

$$A = \begin{bmatrix} -\alpha & \gamma \\ 0 & -\beta \end{bmatrix} \quad \alpha, \beta, \gamma > 0. \tag{11}$$

The following two cases will be considered

Case 1. The matrix (11) has distinct eigenvalues $\lambda_1 = -\alpha$, $\lambda_2 = -\beta$ ($\alpha \neq \beta$)

Case 2. The matrix (11) has one double eigenvalue $\lambda_1 = \lambda_2 = -\alpha$ ($\alpha = \beta$)

In case 1 we have

$$\begin{aligned} x(t) &= e^{At}x_0 = (Z_1 e^{\lambda_1 t} + Z_2 e^{\lambda_2 t})x_0 \\ &= \left(\begin{bmatrix} 1 & \dfrac{\gamma}{\beta-\alpha} \\ 0 & 0 \end{bmatrix} e^{-\alpha t} + \begin{bmatrix} 0 & \dfrac{\gamma}{\alpha-\beta} \\ 0 & 1 \end{bmatrix} e^{-\beta t} \right) x_0 \\ &= \begin{bmatrix} \left(x_{10} + \dfrac{\gamma}{\beta-\alpha}x_{20}\right)e^{-\alpha t} + \dfrac{\gamma}{\alpha-\beta}x_{20}e^{-\beta t} \\ x_{20}e^{-\beta t} \end{bmatrix} \end{aligned}$$

where $x_0 = [x_{10} \quad x_{20}]^T$

and

$$\begin{aligned} \rho(x,t) &= \left[\left(\left(x_{10} + \frac{\gamma}{\beta-\alpha}x_{20}\right)e^{-\alpha t} \right.\right. \\ &\left.\left. + \frac{\gamma}{\alpha-\beta}x_{20}e^{-\beta t}\right)^2 + \left(x_{20}e^{-\beta t}\right)^2\right]^{\frac{1}{2}} \\ &= \begin{cases} \left[\left(\left(x_{10} + \dfrac{\gamma}{\beta-\alpha}x_{20}\right) + \dfrac{\lambda}{\alpha-\beta}x_{20}e^{(\alpha-\beta)t}\right)^2 + \left(x_{20}e^{(\alpha-\beta)t}\right)^2\right]^{\frac{1}{2}} e^{-\alpha t} \text{ for } \beta > \alpha \\ \left[\left(\left(x_{10} + \dfrac{\gamma}{\beta-\alpha}x_{20}\right)e^{(\beta-\alpha)t} + \dfrac{\gamma}{\alpha-\beta}x_{20}\right)^2 + x_{20}^2\right]^{\frac{1}{2}} e^{-\beta t} \text{ for } \alpha > \beta \end{cases}. \end{aligned} \tag{12}$$

From (12) it follows that the positive system (4) with (11) is strongly stable since $\rho(x,t)$ satisfies the conditions of definition 1.

In case 2 we have

$$x(t) = e^{At}x_0 = \begin{bmatrix} (x_{20} + \gamma x_{20} t)e^{-\alpha t} \\ x_{20} e^{-\alpha t} \end{bmatrix}$$

and (13)

$$\rho(x,t) = \left[(x_{10} + \gamma x_{20} t)^2 + x_{20}^2 \right]^{\frac{1}{2}} e^{-\alpha t}.$$

From (13) it follows that in this case the system (4) with (11) ($\alpha = \beta$) exhibits overshoots for enough small value of α and big value of γ.
The simulation results are shown in Figs. 1–5.

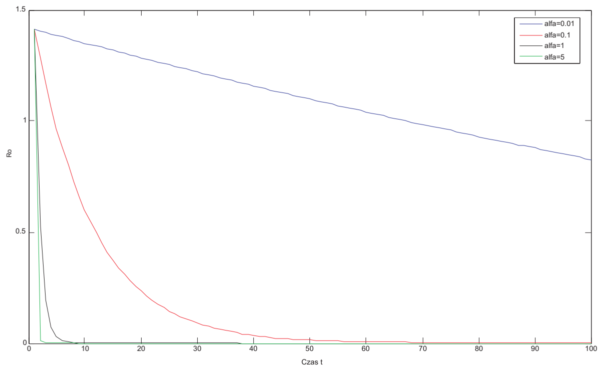

Fig. 1. Simulation for $x_0 = [x_{10}, x_{20}]^T = [1,1]^T$, $\alpha = 0.01;\ 0.1;\ 1\ ;\ 5$; $\gamma = 0.01$, $t \in [0, 100]$

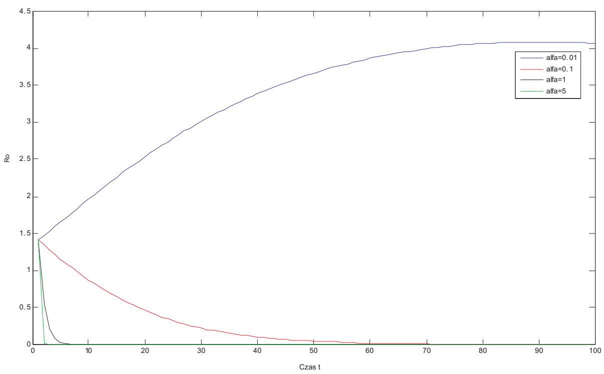

Fig. 2. Simulation for $x_0 = [x_{10}, x_{20}]^T = [1,1]^T$, $\alpha = 0.01;\ 0.1;\ 1\ ;\ 5$; $\gamma = 0.01$, $t \in [0, 100]$

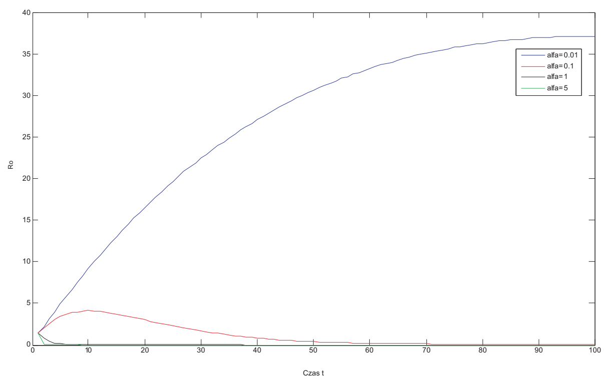

Fig. 3. Simulation for $x_0 = [x_{10}, x_{20}]^T = [1,1]^T$, $\alpha = 0.01;\ 0.1;\ 1\ ;\ 5$; $\gamma = 1$, $t \in [0, 100]$

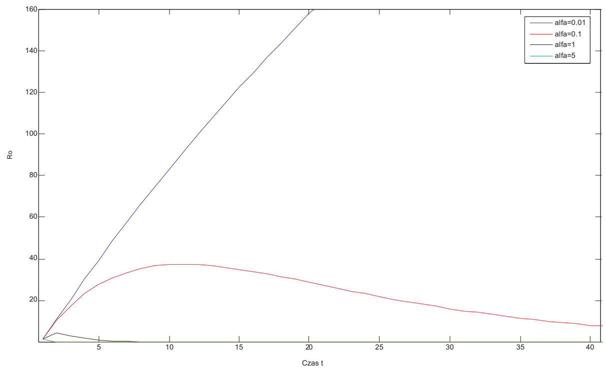

Fig. 4. Simulation for $x_0 = [x_{10}, x_{20}]^T = [1,1]^T$, $\alpha = 0.01;\ 0.1;\ 1\ ;\ 5$; $\gamma = 10$, $t \in [0, 41]$

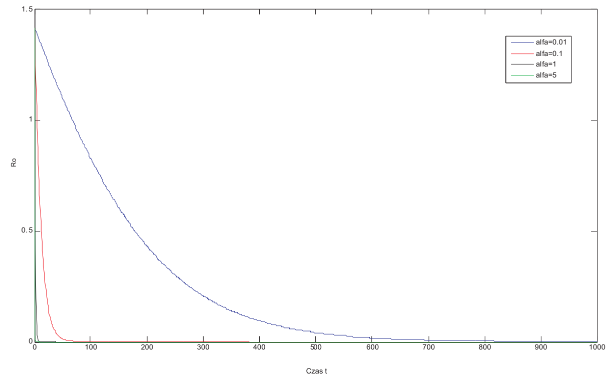

Fig. 5. Simulation for $x_0 = [x_{10}, x_{20}]^T = [1,1]^T$, $\alpha = 0.01;\ 0.1;\ 1\ ;\ 5$; $\gamma = 0.01$, $t \in [0, 41]$

The considerations can be extended for positive discrete-time systems. Consider the positive autonomous discrete-time systems

$$x_{k+1} = Ax_k, \qquad k \in Z_+ \tag{14}$$

where $A \in R_+^{nxn}$.

Definition 2. The system (14) is called strongly stable if for any initial condition $x_0 \in R_+^n$ in the sphere $S(0,r)$ the following conditions are satisfied:

i. $\rho(x_0, k) \leq \rho(x_0, 0)$ for $k \in Z_+$ and $x_0 \in S(0,r)$

ii. $\lim\limits_{k \to \infty} \rho(x_0, k) = 0$

Theorem 4. Let the positive system (14) be asymptotically stable. Then the system is strongly stable if the matrix has distinct eigenvalues.
The proof is similar to the proof of the Theorem 3.

5. Compartmental systems

Consider a compartmental continuous-time system consisting of n compartments. Let $x_i = x_i(t)$, $i = 1, ..., n$, be the amount of a material of the ith compartment. It is assumed that the output flow $F_{ij} \geq 0$ from the jth to the ith compartment ($i \neq j$) depends linearly on x_j, $F_{ij} = f_{ij}x_j$, where the coefficients f_{ij} are independent of x_j and time-invariant.

Let F_{0i} be the output flow of the material from the ith compartment to the environment ($F_{0i} = f_{0i}x_i$, $i = 1, ..., n$) and $u_i = u_i(t)$ be the input flow of the material to the ith compartment from the environment. From the balance of the material of the ith compartment [4] we obtain the state equation

$$\dot{x} = Fx + u \tag{15}$$

where $x = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix}^T$, $u = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix}^T$, $F = [f_{ij}]_{i,j=1,...,n}$ and the sum of entries of every column of the matrix F is not positive, i.e.

$$-f_{ij} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} f_{ij} \geq 0 \text{ and } f_{ij} \geq 0 \quad \text{for } i \neq j \; (i, j = 1, ..., n), \tag{16}$$

The compartmental matrix F is a particular case of the Metzler matrix, since $f_{ij} \geq 0$ for $i \neq j$. Note that if

$$-f_{ij} > \sum_{\substack{i=1 \\ i \neq j}}^{n} f_{ij} \geq 0 \tag{17}$$

then the compartmental system is asymptotically stable. Therefore, we have the following theorem:

Theorem 5. The compartmental system (15) is strongly stable if its matrix F satisfies the condition (17) and has distinct eigenvalues.

Similar results can be obtained for discrete-time linear compartmental systems [4]. The considerations can be extended for 2D compartmental systems [2].

6. Electrical circuits

Consider the electrical circuit shown in Fig. 6. with known resistances R_1, R_2, R_3, capacitances C_1, C_2 and a source voltage $e = e(t)$. The voltages $u_1 = u_1(t)$, $u_2 = u_2(t)$ on the capacitances are chosen as the state variables.

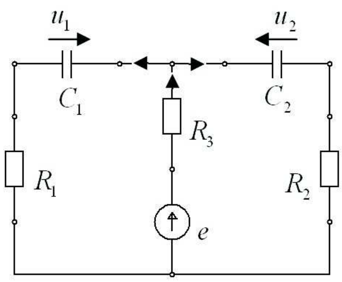

Fig. 6. Electrical circuit

Applying the Kirchhoff laws we may write the equations

$$R_1 C_1 \dot{u}_1 + u_1 + R_3(C_1 \dot{u}_1 + C_2 \dot{u}_2) = e$$
$$R_3(C_1 \dot{u}_1 + C_2 \dot{u}_2) + u_2 + R_2 C_2 \dot{u}_2 = e$$

which can be written in the form

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be \tag{18}$$

where

$$A = \begin{bmatrix} -\frac{R_2 + R_3}{C_1[R_1(R_2 + R_3) + R_2R_3]} & \frac{R_3}{C_1[R_1(R_2 + R_3) + R_2R_3]} \\ \frac{R_3}{C_2[R_1(R_2 + R_3) + R_2R_3]} & \frac{R_1 + R_3}{C_2[R_1(R_2 + R_3) + R_2R_3]} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{R_2}{C_1[R_1(R_2 + R_3) + R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2 + R_3) + R_2R_3]} \end{bmatrix} \tag{19}$$

From (19) it follows that A is the Metzler matrix and B has positive entries. Therefore, the electrical circuit is an example of continuous-time positive system.

Note that the matrix A can be written as a product

$$A = D\bar{A} \tag{20}$$

of the nonsingular diagonal matrix

$$D = diag \begin{bmatrix} \frac{1}{C_1} & \frac{1}{C_2} \end{bmatrix} \tag{21}$$

and the symmetric matrix

$$\bar{A} = \begin{bmatrix} -\frac{R_2 + R_3}{R_1(R_2 + R_3) + R_2R_3} & \frac{R_2}{R_1(R_2 + R_3) + R_2R_3} \\ \frac{R_3}{R_1(R_2 + R_3) + R_2R_3} & -\frac{R_1 + R_3}{R_1(R_2 + R_3) + R_2R_3} \end{bmatrix} \tag{22}$$

In the sequel the following lemma will be used.

Lemma. Let $A = A^T \in R^{n \times n}$, $D = diag \begin{bmatrix} d_1 & d_2 & ... & d_n \end{bmatrix}$, $d_i \neq 0$, $i = 1, \ldots, n$ and

$$A_1 = D^{-1}A, \qquad A_2 = AD^{-1}. \tag{23}$$

Then both matrices A_1 and A_2 have the same real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ which are related with the eigenvalues $\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n$ of the matrix A by

$$\bar{\lambda}_i = d_i \lambda_i, \qquad i = 1, ..., n \tag{24}$$

Proof. The eigenvalues $\bar{\lambda}_i$, $i = 1, ..., n$ are real since by assumption A is symmetric. From (23) we have

$$\det[I_n\lambda - A_1] = \det[I_n\lambda - D^{-1}A] = \det[D^{-1}(D\lambda - A]$$
$$= \det D^{-1} \det[I_n\bar{\lambda} - A]$$

where

$$I_n\bar{\lambda} = D\lambda. \tag{25}$$

The equality (25) implied the relation (24).
Note that

$$A_1^T = (D^{-1}A)^T = A^T D^{-1} = AD^{-1} = A_2$$

Hence A_1 and A_2 have the same eigenvalues since the matrices A and A^T have the same spectrum.
Applying this lemma to the matrix A defined by (19) we obtain:

Let $\bar{s}_1$ and $\bar{s}_2$ be the real eigenvalues of the symmetric matrix (22). Then the eigenvalues s_1, s_2 of the matrix A are also real and are given by the equalities

$$s_1 = C_1\bar{s}_s, \qquad s_2 = C_2\bar{s}_2$$

In the general case of electrical circuit with n known capacitors C_1,C_2,...,C_n resistors and source voltages we obtain the matrix A which can be written as the product $A = D\bar{A}$ of the nonsingular diagonal matrix

$$D = diag\begin{bmatrix} \frac{1}{C_1} & \frac{1}{C_2} & ... & \frac{1}{C_n} \end{bmatrix} \tag{26}$$

and the symmetric matrix

$$\bar{A} = \begin{bmatrix} -R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & -R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & -R_{nn} \end{bmatrix}, \quad R_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{n} R_{ij}, \quad i = 1, ..., n. \tag{27}$$

It is well-known that the matrix (27) is negative definite and has only negative real eigenvalues $\bar{s}_1, \bar{s}_2, ..., \bar{s}_n$. From (24) it follows that matrix has also only negative eigenvalues $s_1, s_2, ..., s_n$ given by

$$s_i = C_i\bar{s}_i, \qquad i = 1, ..., n. \tag{28}$$

Therefore, the matrix A is negative definite and the following theorem has been proved.

Theorem 6. Electrical circuits consisting of resistances, capacitances and source voltages are strongly stable.
A dual theorem is valid for electrical circuits consisting of resistances, inductances and source voltages.

7. Concluding remarks

The concept of strong stability has been extended for positive and compartmental linear systems. It has been shown that: 1) the asymptotically stable positive and compartmental systems are strongly stable if the eigenvalues of the system matrix are distinct, 2) electrical circuits consisting of resistances, capacitances (inductances) and source voltages are strongly stable. The considerations can be extended for positive and compartmental 2D linear systems.

REFERENCES

[1] L. Farina and S. Rinald, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.

[2] T. Kaczorek, *Positive 1D and 2D Systems*, Springer Verlag, London, 2001.

[3] T. Kaczorek, "2D compartmental linear systems", *Proc. Conf. Computer Applications in Electrotechnics*, 93–99 (2004).

[4] T. Kaczorek, "What is the relationship between the singular compartmental systems and the electrical circuits", *Electrotechnical Review* R8 (4), 316–324 (2004).

[5] N. Karcanias, G. Halikies, and A. Papageorgini, "Strong stability of internal system descriptions", *Proc. European Control Conf.*, 1134–1141 (2007).

[6] T. Kaczorek, *Vectors and Matrices in Automation and Electrotechnics*, Wydawnictwo Naukowo-Techniczne, Warszawa, 1998, (in Polish).

[7] T. Kaczorek, "Electrical circuits as examples of positive singular continuous-time systems", *Proc. SPETO 98*, 37–43 (1998).