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On the solution of the implicit Roesser model

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Abstract. The main objective of this work is to provide a closed formula for the backward and symmetric solution of the 2-D implicit Roesser model. The relative forward and backward fundamental matrix is of fundamental importance in our approach. An algorithm for the determination of the backward fundamental matrix sequense is also given.

Key words: Roesser model, forward solution, backward solution, symmetric solution, implicit 2-D systems, fundamental matrix sequence.

1. Introduction

In recent years, the field of multidimensional linear systems theory has attracted many researchers [1–3] due to the wide applications in areas such as image processing, linear multipass processes, iterative learning control systems, lumped and distributed networks e.t.c.. The Roesser [4] and the Fornasini-Marchesini [5] 2-D state-space models have been proven useful in such areas (see also [3]). A main disadvantage of these models is that they require causality or a milder notion like recursibility. However, in the 2-D plane there is no natural notion of causality, if we think, for example of the discretized version of the hyperbolic equation [6] or the heat equation which is a two-variable partial differential equation with boundary conditions specified on all sides of a planar region or the long transmission line where the voltage at a point depends on the voltage on either side of a point. Other examples are also in image processing where the 2-D system may have right to left dependencies as well as left to right dependencies. To overcome the problem of causality and recursibility, implicit models have been proposed. More specifically, [7] has proposed a general singular model (GSM) or otherwise called implicit Fornasini-Marchesini model, while [8] and [9] have proposed a special case of the general singular model, the implicit Roesser model. However, there is a number of important cases where singularity in the resulting model structure can be avoided by using appropriate analysis tools i.e. linear repetitive processes [10]. However, this is not the case in this paper.

In [11] and [12] a forward solution to the 2-D GSM and the implicit Roesser model respectively, was investigated in terms of the forward fundamental matrix of the system. In both cases, it is found the semistate sequence is given the inputs and the initial semistate value. However, certain questions still remain as concerns: a) the symmetric solution, where the inputs and the boundary values are prescribed, and b) the backward solution, where the inputs and the final values are prescribed. In this note, we provide analytic solutions for the backward and symmetric solution and thus extend in this way the results presented in [13] for the case of one-variable descriptor systems.

2. Background

Consider the singular dynamical system of equations

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

with $x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p, k = 0, 1, ..., N-1$. The interval of interest of index k is $[0, N]$, with $u(k)$ nonzero for $k = 0, 1, .., N$. By assuming that the pencil $zE - A$ is regular i.e. $\det(z_0 E - A) \neq 0$ for some $z_0 \in \mathbb{C}$, then for some $R > 0$ and $|z| > R$, the Laurent series expansion about infinity for the resolvent matrix is given by

$$(zE - A)^{-1} = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i} \tag{2}$$

where μ is the index of nilpotence and the sequence Φ_i is known as the (forward) fundamental matrix. Similarly for some $R > 0$ and for $0 < |z| < R$, the Laurent series expansion about zero for the resolvent matrix is given by

$$(zE - A)^{-1} = \sum_{i=-p}^{\infty} F_{-i} z^i \tag{3}$$

where the sequence F_{-i} is known as the (backward) fundamental matrix. Explicit formulas for the coefficients Φ_i has been given in [14–18]. There have been several interpretations of Eq. (1). From a dynamical standpoint we may consider that the initial condition $x(0)$ is given and that is desired to determine the state $x(k)$ in a forward fashion from the input sequence and the previous values of the semistate. We call this the forward solution of (1) and is given by [13]:

$$x(k) = \Phi_k E x(0) + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u(i) \tag{4}$$

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A variant of this is to consider $x(N)$ as given and then determine $x(k)$ in a backward fashion from the input and future values of the semistate. We call this the backward solution of (1) and is given by [13]:

$$x(k) = -F_{k-N-1}Ex(N) + \sum_{i=k-p}^{N-1} F_{k-i}Bu(i). \quad (5)$$

Another interpretation, arising in economics (where k might not be the time variable) and elsewhere, is to determine the semistate $x(k)$ for intermediate values of k, given the sequence $\{u(k)\}$ and admissible $x(0)$ and $x(N)$. We call this the symmetric solution of (1) and it is given by [13]:

$$x(k) = \Phi_k Ex(0) - \Phi_{-N+k}Ex(N) + \sum_{i=0}^{N-1} \Phi_{k-i-1}Bu(i) \quad (6)$$

where $x(0)$,$x(N)$ satisfy a symmetric boundary condition of the form

$$W_0 x(0) + W_N x(N) = w \in \mathbb{R}^n \quad (7)$$

and $\begin{bmatrix} W_0 & W_N \end{bmatrix}$ is a prescribed real matrix with full rank n, and w a real vector.

Consider the 2-D linear discrete time system proposed in [7] as a generalization of the 2-D state-space model given in [19]

$$\begin{aligned} Ex(i+1,j+1) &= A_0 x(i,j) + A_1 x(i+1,j) \\ &+ A_2 x(i,j+1) + B_0 u(i,j) \\ &+ B_1 u(i+1,j) + B_2 u(i,j+1) \end{aligned} \quad (8)$$

where i, j are integer-value vertical and horizontal coordinates, respectively, $x(i,j) \in \mathbb{R}^n$ is the partial state vector at (i,j), $u(i,j) \in \mathbb{R}^m$ is the input vector, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}, k = 0,1,2$ and matrices $E, A_0 \in \mathbb{R}^{n\times n}$ exists and are not necessarily nonsingular. This model includes similar generalization of other 2-D state space models such as the Fornasini and Marchesini [5] and the Roesser 2-D model [23]. If $E \neq I$ we call these models implicit 2-D systems. We shall call (8) the general singular model (GSM) or otherwise the implicit Fornasini-Marchesini model. If E is non-square or $\det(E) = 0$ we call these models singular 2-D systems. One particular case of (8) is the implicit Roesser model proposed in [8] and [9] as a generalization of the Roesser 2-D model given in [4]

$$\begin{aligned} &\underbrace{\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}}_{\tilde{x}(i,j)} \\ &= \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}}_{x(i,j)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u(i,j) \end{aligned} \quad (9)$$

where $x^h(i,j) \in \mathbb{R}^{t_1}$ and $x^v(i,j) \in \mathbb{R}^{t_2}$ (with $t_1 + t_2 = n$) denote the so-called horizontal and vertical partial state vectors. It is shown in [20] that the implicit Roesser and the implicit FM model are equivalent. Define for example

$$E_I = \begin{bmatrix} E_1 & 0 \\ E_3 & 0 \end{bmatrix}, \quad E_{II} = \begin{bmatrix} 0 & E_2 \\ 0 & E_4 \end{bmatrix}.$$

Then (9) may be rewritten as

$$E_I x(i+1,j) + E_{II} x(i,j+1) = Ax(i,j) + Bu(i,j) \quad (10)$$

a special case of (8). Due to the equivalence of the above models we consider in the rest of the paper only the Roesser model, due to its simplest form. An extensive study of implicit 2-D systems is given in [3].

According to [21] there are various ways to specify the boundary conditions (BCs) and the region of interest for the implicit FM and Roesser models:

a) First suppose that the 2-D implicit system has BCs specified along the $i-$ and $j-$ axes. For the Roesser model this means we know:

$$\begin{aligned} x(i,0) &= x_{i0}, i = 0,1,...,N \\ x(0,j) &= x_{0j}, j = 0,1,...,M \end{aligned} \quad (11)$$

where x_{i0} and x_{0j} are known vectors. Then, if the region of interest is the rectangle $[0,N]\times[0,M]$ in the (i,j) – plane, we are concerned with finding what could be called a "forward solution".

b) If the BCs are specified along the upper and right-hand sides of the rectangle:

$$\begin{aligned} x(i,M) &= x_{iM}, i = 0,1,...,N \\ x(N,j) &= x_{Nj}, j = 0,1,...,M \end{aligned} \quad (12)$$

then the solution on $[0,N]\times[0,M]$ could be called "backward solution".

c) A general case which includes both of these situations is where the BCs are of the split or two-point form:

$$\begin{aligned} C^u_{i,0} x(i,0) + C^u_{i,M} x(i,M) &= c^u_i, \quad 0 \le i \le N \\ C^h_{0,j} x(0,j) + C^h_{N,j} x(N,j) &= c^h_i, \quad 0 \le j \le M \end{aligned} \quad (13)$$

with $\begin{bmatrix} C^u_{i,0} & C^u_{i,M} \end{bmatrix}$ and $\begin{bmatrix} C^h_{0,j} & C^h_{N,j} \end{bmatrix}$ prescribed matrices of full row rank and c^u_i, c^h_i given vectors. If the BCs are of the split form given above or otherwise involve the semistate along all boundaries of the rectangular region $[0,N]\times[0,M]$ then the solution on $[0,N]\times[0,M]$ could be called "symmetric solution".

An example of the implicit Roesser model is given by the 2-D realization of a nonrecursible mask in digital image processing [12]. Implicit Roesser models are also arising from the discretization of continuous-time systems that are described by partial differential equations i.e. the standard discretization of the elliptic equation that results in a five-point discrete mask or the discretization of the diffusion equation that results in a four-point discrete mask [22]. Even in the case where the continuous time model is described by a standard 2-D state space model, the discretization method employed produces a singular discrete approximation [10]. However, in some special cases, it is possible to avoid the disadvantage by applying transformation techniques or alternative discretization methods [10].

Assuming that the polynomial matrix

$$G(z_1, z_2) = z_1 E_I + z_2 E_{II} - A \tag{14}$$

and the Laurent expansion at infinity of $G(z_1, z_2)^{-1}$ exists, is unique [12,23], and is given by:

$$G(z_1, z_2)^{-1} = \sum_{i=-n_1}^{\infty} \sum_{j=-n_2}^{\infty} T_{i,j} z_1^{-i} z_2^{-j} \tag{15}$$
$$(n_1 \le n, n_2 \le n) \quad \text{and} \quad |z_1| > \sigma_1 > 0, |z_2| > \sigma_2 > 0$$

where the matrix sequence $\{T_{i,j}\}$ is known as the forward fundamental matrix. Note that a necessary and sufficient condition for the uniqueness of the fundamental matrix sequence $\{T_{i,j}\}$ is that condition $\deg_z |G(z,z)| = \deg_{z_1} |G(z_1, z_2)| + \deg_{z_2} |G(z_1, z_2)|$ is satisfied [12,23], where $\deg_{z_i} |G(z_1, z_2)|$ is the degree of $\det G(z_1, z_2)$ in z_i, with $i = 1, 2$ and $\deg_z |G(z,z)|$ is the degree of $\det G(z,z)$ in z. The finite lower limits on the summation permit us the introduction of a unique transition matrix in the same way as in the standard case [7]. If the determinant of $G(z_1, z_2)$ has the above characteristic property, then it is called principal, and the corresponding system is also called principal [21]. Thus, principal systems consist a particular class of singular systems.

Assuming now that the Laurent expansion at zero of $G(z_1, z_2)^{-1}$ exists, is unique and is given by

$$G(z_1, z_2)^{-1} = \sum_{i=-\infty}^{\ell_1} \sum_{j=-\infty}^{\ell_2} V_{i,j} z_1^{-i} z_2^{-j} \quad |z_1| < \sigma_1, |z_2| < \sigma_2 \tag{16}$$

where the matrix sequence $\{V_{i,j}\}$ is known as the backward fundamental matrix. We shall propose in this paper a necessary and sufficient condition for the uniqueness of the backward fundamental matrix sequence, in terms of the least degree of $|G(z_1, z_2)|$ in z_i and the least degree of $|G(z,z)|$ in z. A generalized Leverrier technique for computing the forward fundamental matrix sequence is available [23,24], so that we may assume that this matrix sequence is given. An algorithm for the computation of the backward fundamental matrix is proposed in Section 3, either by using the forward fundamental matrix of the inverse of the dual polynomial matrix $\tilde{G}(z_1, z_2) = z_1 z_2 G\left(\frac{1}{z_1}, \frac{1}{z_2}\right) = z_2 E_I + z_1 E_{II} - A z_1 z_2$ of $G(z_1, z_2)$ or directly in terms of the coefficient matrices of the adjoint matrix of $G^{-1}(z_1, z_2)$ and the coefficients of the determinant of $G(z_1, z_2)$ [12].

A forward solution to the 2-D GSM and the implicit Roesser model respectively, in terms of the forward fundamental matrix sequence $\{T_{i,j}\}$ have been proposed in [11,12]. Following similar methods to those of [13], we produce in Section 4 a closed formula for the backward and symmetric solution of the implicit Roesser model (9) in terms of the forward fundamental matrix sequence $\{T_{i,j}\}$ and backward fundamental sequence $\{V_{i,j}\}$ of $G(z_1, z_2)$.

3. Computation of the backward fundamental matrix sequence of a two-variable polynomial matrix

In [12] the inverse of the polynomial matrix $G(z_1, z_2) = z_1 E_I + z_2 E_{II} - A$ has been obtained by

$$(z_1 E_I + z_2 E_{II} - A)^{-1} = \frac{R(z_1, z_2)}{d(z_1, z_2)} \tag{17}$$

where

$$R(z_1, z_2) = \sum_{i=f_1^d}^{f_1^u} \sum_{j=f_2^d}^{f_2^u} R_{i,j} z_1^i z_2^j, \qquad f_i^u = \deg_{z_i} R(z_1, z_2), i = 1, 2 \tag{18}$$

$$d(z_1, z_2) = \sum_{i=d_1^d}^{d_1^u} \sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^i z_2^j, \qquad d_i^u = \deg_{z_i} d(z_1, z_2), i = 1, 2 \tag{19}$$

where f_i^d and d_i^d are the lower degrees of $R(z_1, z_2)$ and $d(z_1, z_2)$ respectively in z_i. Then we have that

$$\underbrace{\left(\sum_{i=d_1^d}^{d_1^u} \sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^i z_2^j\right)}_{d(z_1,z_2)} \underbrace{\left(\sum_{i=-\infty}^{\ell_1} \sum_{j=-\infty}^{\ell_2} V_{i,j} z_1^{-i} z_2^{-j}\right)}_{(z_1 E_1 + z_2 E_{11} - A)^{-1}} = \underbrace{\sum_{i=f_1^d}^{f_1^u} \sum_{j=f_2^d}^{f_2^u} R_{i,j} z_1^i z_2^j}_{R(z_1,z_2)} \tag{20}$$

or equivalently, by replacing the indices i, j on the backward fundamental matrix sequences $V_{i,j}$ by $-i, -j$, we get

$$\underbrace{\left(\sum_{i=d_1^d}^{d_1^u} \sum_{j=d_2^d}^{d_2^u} d_{i,j} z_1^i z_2^j\right)}_{d(z_1,z_2)} \underbrace{\left(\sum_{i=-\ell_1}^{\infty} \sum_{j=-\ell_2}^{\infty} V_{-i,-j} z_1^{i} z_2^{j}\right)}_{(z_1 E_1 + z_2 E_{11} - A)^{-1}} = \underbrace{\sum_{i=f_1^d}^{f_1^u} \sum_{j=f_2^d}^{f_2^u} R_{i,j} z_1^i z_2^j}_{R(z_1,z_2)}$$

and thus $f_i^d = -\ell_i + d_i^d, i = 1, 2$ (by equating the lowest degrees of both sides). By equating the coefficient matrices of the corresponding powers of $z_1^i z_2^j$, on both sides of the resulting equation (first for the pairs (i, j) where $f_1^d \le i \le f_1^u$

and $f_2^d \leq j \leq f_2^u$, since then $R_{i,j}$ is not necessarily zero and then for all other pairs of (i,j) since then $R_{i,j}=0$), yields

$$R_{i,j} = \sum_{l=d_1^d}^{d_1^u} \sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{l-i,m-j}, \quad (f_1^d \leq i \leq f_1^u \text{ and } f_2^d \leq j \leq f_2^u) \tag{21}$$

$$0 = \sum_{l=d_1^d}^{d_1^u} \sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{l-i,m-j} \quad \text{for every other pair } (i,j) \tag{22}$$

which allows the computation of $V_{i,j}$ in the stated region in terms of its values for smaller i,j. Thus (22) constitutes another form of the Cayley Hamilton theorem for the 2-D matrix pencils. In the case where $d_{d_1^d,d_2^d}=0$, then the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ may not be unique as we can see in the following Theorem.

Theorem 1. Suppose that $d^d = d_1^d + d_2^d$, where d^d is the less degree in z of $\det G(z,z) = \det(zE_I + zE_{II} - A)$ and d_i^d are the least degrees of $d(z_1,z_2)$ in z_i, or equivalently that $d_{d_1^d,d_2^d} \neq 0$. Then the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ is unique.

Proof. Let

$$\tilde{G}(z_1,z_2) \equiv z_1 z_2 G\left(\frac{1}{z_1},\frac{1}{z_2}\right) = z_1 E_{II} + z_2 E_I - A z_1 z_2. \tag{23}$$

Since

$$\tilde{d}(z_1,z_2) = \det\left[\tilde{G}(z_1,z_2)\right] = \det\left[z_1 z_2 G\left(\frac{1}{z_1},\frac{1}{z_2}\right)\right] = $$
$$= z_1^n z_2^n \left(\sum_{i=d_1^d}^{d_1^u} \sum_{i=d_2^d}^{d_2^u} d_{i,j} z_1^{-i} z_2^{-j}\right) = \sum_{i=d_1^d}^{d_1^u} \sum_{i=d_2^d}^{d_2^u} d_{i,j} z_1^{n-i} z_2^{n-j}$$

the above condition is equivalent to the condition $\tilde{d} = \tilde{d}_1 + \tilde{d}_2$ where $\tilde{d}_i = \deg_{z_i} |z_1 E_{II} + z_2 E_I - A z_1 z_2|, i = 1,2$ and $\tilde{d} = \deg_z |zE_{II} + zE_I - A z_1 z_2|$. The proof of this Theorem follows from the unique construction of an explicit formula for the computation of the Laurent expansion at zero of $G(z_1,z_2)^{-1}$. Equation (21) may be rewritten as

$$\underbrace{\begin{bmatrix} d_{d_1^d,d_2^d} I_n & 0 & & & & 0 \\ d_{d_1^d+1,d_2^d} I_n & d_{d_1^d,d_2^d} I_n & & & & \\ \vdots & \vdots & \ddots & & & \\ d_{d_1^u,d_2^d} I_n & d_{d_1^u-1,d_2^d} I_n & \ddots & \ddots & & \\ 0 & d_{d_1^u,d_2^d} I_n & & \ddots & \ddots & \\ 0 & & & & \ddots & \ddots \\ & & & d_{d_1^u,d_2^d} I_n & \cdots & d_{d_1^d,d_2^d} I_n \end{bmatrix}}_{P_{d_2^d}}$$

$$\underbrace{\begin{bmatrix} V_{\ell_1,\ell_2} \\ V_{\ell_1-1,\ell_2} \\ \vdots \\ V_{\ell_1-(f_1^u-f_1^d),\ell_2} \end{bmatrix}}_{V_{\ell_2}} = \underbrace{\begin{bmatrix} R_{f_1^d,f_2^d} \\ R_{f_1^d+1,f_2^d} \\ \vdots \\ R_{f_1^u,f_2^d} \end{bmatrix}}_{R_{f_2^d}}$$

and

$$\underbrace{\begin{bmatrix} P_{d_2^d} I_n & 0 & & & & 0 \\ P_{d_2^d+1} I_n & P_{d_2^d} I_n & & & & \\ \vdots & \vdots & \ddots & & & \\ P_{d_2^u} I_n & P_{d_2^u-1} I_n & \ddots & \ddots & & \\ 0 & P_{d_2^u} I_n & & \ddots & \ddots & \\ 0 & & & & \ddots & \ddots \\ & & & P_{d_2^u} I_n & \cdots & P_{d_2^d} I_n \end{bmatrix}}_{P}$$

$$\underbrace{\begin{bmatrix} V_{\ell_2} \\ V_{\ell_2-1} \\ \vdots \\ V_{\ell_2-(f_2^u-f_2^d)} \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} R_{f_2^d} \\ R_{f_2^d+1} \\ \vdots \\ R_{f_2^u} \end{bmatrix}}_{R}$$

where

$$P_i = \begin{bmatrix} d_{d_1^d,i} I_n & 0 & & & & 0 \\ d_{d_1^d+1,i} I_n & d_{d_1^d,i} I_n & & & & \\ \vdots & \vdots & \ddots & & & \\ d_{d_1^u,i} I_n & d_{d_1^u-1,i} I_n & \ddots & \ddots & & \\ 0 & d_{d_1^u,i} I_n & & \ddots & \ddots & \\ 0 & & & & \ddots & \ddots \\ & & & d_{d_1^u,i} I_n & \cdots & d_{d_1^d,i} I_n \end{bmatrix}$$

$$i = d_2^d, d_2^d+1, \ldots, d_2^u$$

$$V_i = \begin{bmatrix} V_{\ell_1,i} \\ V_{\ell_1-1,i} \\ \vdots \\ V_{\ell_1-(f_1^u-f_1^d),i} \end{bmatrix}, i = \ell_2, \ell_2-1, ..., \ell_2-(f_2^u-f_2^d)$$

and

$$R_i = \begin{bmatrix} R_{f_1^d,i} \\ R_{f_1^d+1,i} \\ \vdots \\ R_{f_1^u,i} \end{bmatrix}, i = f_2^d, f_2^d+1, ..., f_2^u$$

Due to the special Toeplitz form of $P_{d_2^d}$, we find that the unique (i.e. $\det P_{d_2^d} \neq 0$) inverse of $P_{d_2^d}$ is

$$D = P_{d_2^d}^{-1} = \begin{bmatrix} r_0 I_n & & & 0 \\ r_1 I_n & r_0 I_n & & \\ \vdots & \vdots & \ddots & \vdots \\ r_{f_1^u-f_1^d} I_n & r_{f_1^u-f_1^d-1} I_n & \cdots & r_0 I_n \end{bmatrix}$$

where

$$r_0 = \frac{1}{d_{d_1^d,d_2^d}}$$

and

$$r_j = (-1)^j \left(\frac{1}{d_{d_1^d,d_2^d}} \right)^{j+1}$$

$$\det \begin{bmatrix} d_{d_1^d+1,d_2^d} & d_{d_1^d+2,d_2^d} & \cdots & d_{d_1^d+j,d_2^d} \\ d_{d_1^d,d_2^d} & d_{d_1^d+1,d_2^d} & \cdots & d_{d_1^d+j-1,d_2^d} \\ & d_{d_1^d,d_2^d} & \cdots & d_{d_1^d+2,d_2^d+j-2} \\ & & \ddots & \vdots \\ 0 & & d_{d_1^d,d_2^d} & d_{d_1^d+1,d_2^d} \end{bmatrix}$$

or equivalently

$$r_j = -\frac{1}{d_{d_1^d,d_2^d}} \sum_{i=0}^{j-1} \left[d_{d_1^d+j-i,d_2^d} \times r_i \right], \quad j = 1, 2, ..., f_1^u - f_1^d$$

so that we may write for the elements of V_{i,ℓ_2} the expressions

$$V_{i,\ell_2} = \sum_{j=0}^{\ell_1-i} r_j R_{d_i^d-i-j,d_2^d}, \quad i = \ell_1-(f_1^u-f_1^d), .., \ell_1-1, \ell_1. \tag{24}$$

Due to the special Toeplitz form of P, we find also that the unique P^{-1} is

$$P^{-1} = \begin{bmatrix} D_0 I_n & & & 0 \\ D_1 I_n & D_0 I_n & & \\ \vdots & \vdots & \ddots & \vdots \\ D_{(f_2^u-f_2^d)} I_n & D_{f_2^u-f_2^d-1} I_n & \cdots & D_0 I_n \end{bmatrix}$$

where

$$D_0 = P_{f_2^d}^{-1}$$

and

$$D_i = -\left(\sum_{j=0}^{i-1} D_j P_{f_2^d-j+i} \right) P_{f_2^d}^{-1}, i = 1, 2, ..., (f_2^u - f_2^d).$$

Thus

$$V_i = \sum_{j=0}^{\ell_2-i} D_j R_{d_2^d-i-j}, i = \ell_2-(f_2^u-f_2^d), .., \ell_2-1, \ell_2.$$

For the calculation of $V_{i,j}$ for less values of i and/or j we rewrite (22) as

$$\begin{gathered} 0 = \sum_{l=d_1^d}^{d_1^u} \sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{-i+l,-j+m} \Longrightarrow \\ V_{-i,-j} = \frac{1}{d_{d_1^d,d_2^d}} \sum_{l=d_1^d}^{d_1^u} \sum_{m=d_2^d}^{d_2^u} d_{l,m} V_{-i+l,-j+m} \\ (l,m) \neq \left(d_1^d, d_2^d\right) \end{gathered} \tag{25}$$

From (24) and (25) we obtain a unique form of the Laurent expansion of $G(z_1,z_2)^{-1}$ and thus the Theorem has been proved.

Since (25) allows the computation of $V_{i,j}$ in the stated region, it constitutes the Cayley-Hamilton theorem for the 2-D singular system (9) in terms of the backward fundamental matrix sequence. Condition $d^d = d_1^d + d_2^d$ is nothing but the requirement that the least degrees of z_1 and z_2 in $|G(z_1,z_2)|$ both appear in the same term. Systems (9) satisfying the above condition constitute a particular class of implicit Roesser models that is both nonempty and potentially interesting i.e. non-recursible masks (see example at the end of the paper). We shall call such systems co-principal.

The Laurent expansion about zero of $G(z_1,z_2)^{-1}$ given in (16) is related with the Laurent expansion at infinity given in (15) of the inverse of the dual matrix $\tilde{G}(z_1,z_2)$ as we can see in the following Lemma.

Lemma 2. Let the Laurent expansion at infinity of $\tilde{G}(z_1,z_2)^{-1}$ be

$$\tilde{G}(z_1,z_2)^{-1} = \sum_{p=-f_1}^{\infty} \sum_{q=-f_2}^{\infty} \tilde{T}_{p,q} z_1^{-p} z_2^{-q} \tag{26}$$

and (16) be the Laurent expansion at zero of $G(z_1,z_2)^{-1}$. Then

$$f_i+1=\ell_i \text{ and } V_{-i,-j}=\tilde{T}_{i+1,j+1}$$
$$i=\ell_1,\ell_1-1,.... \text{ and } j=\ell_2,\ell_2-1,.... \tag{27}$$

Proof. We have that

$$\begin{aligned}
G(z_1,z_2)&=z_1z_2\tilde{G}\left(\frac{1}{z_1},\frac{1}{z_2}\right)\Leftrightarrow\\
G(z_1,z_2)^{-1}&=z_1^{-1}z_2^{-1}\tilde{G}\left(\frac{1}{z_1},\frac{1}{z_2}\right)^{-1}\\
&=z_1^{-1}z_2^{-1}\sum_{p=-f_1}^{\infty}\sum_{q=-f_2}^{\infty}\tilde{T}_{p,q}\left(\frac{1}{z_1}\right)^{-p}\left(\frac{1}{z_2}\right)^{-q}\\
&=\sum_{p=-f_1}^{\infty}\sum_{q=-f_2}^{\infty}\tilde{T}_{p,q}z_1^{p-1}z_2^{q-1}\\
&\equiv\sum_{p_1=-\infty}^{\ell_1}\sum_{q_1=-\infty}^{\ell_2}V_{p_1,q_1}z_1^{-p_1}z_2^{-q_1}\\
&\overset{\substack{q_1=-q+1\\p_1=-p+1}}{=}\sum_{p=-\ell_1+1}^{\infty}\sum_{q=-\ell_2+1}^{\infty}V_{-p+1,-q+1}z_1^{p-1}z_2^{q-1}.
\end{aligned}\tag{28}$$

Equating the coefficients of the powers of $z_i, i=1,2$ we obtain the proof of Lemma.

We conclude from the above Lemma that the Laurent expansion at zero of $G(z_1,z_2)^{-1}$ exists and is unique iff the Laurent expansion at infinity of $\tilde{G}(z_1,z_2)^{-1}$ exists and is unique or otherwise when $\tilde{d}^d=\tilde{d}_1+\tilde{d}_2$,where $\tilde{d}_i=\deg_{z_i}|z_1E_{II}+z_2E_I-Az_1z_2|, i=1,2$ and $\tilde{d}^d=\deg_z|zE_{II}+zE_I-Az_1z_2|$. A direct result of Lemma 2 is that the Leverrier algorithm presented in [12,23] may also be used for the computation of both the forward and backward fundamental matrix sequence. Therefore, Lemma 2 give us an alternative method from the algorithm presented in Theorem 1 for the computation of the backward fundamental matrix sequence.

4. Solutions of the implicit Roesser model

An interesting result that connects the solutions of (10) and the ones of the dual 2-D implicit Roesser model

$$\begin{aligned}&E_I\tilde{x}(i,j+1)+E_{II}\tilde{x}(i+1,j)\\&=A\tilde{x}(i+1,j+1)+B\tilde{u}(i+1,j+1)\end{aligned}\tag{29}$$

in the closed interval $[0,N]\times[0,M]$ is given by the following Lemma.

Lemma 3. (a) If $\tilde{x}(i,j)$ is a solution of (29) for the non-zero input $\tilde{u}(i,j)$, then the sequence $x(i,j)=\tilde{x}(N-i,M-j)$ is a solution of the dual Eq. (10) for the nonzero input $u(i,j)=\tilde{u}(N-i,M-j)$.

(b) If $x(i,j)$ is a solution of (10) for the non-zero input $u(i,j)$, then the sequence $\tilde{x}(i,j)=x(N-i,M-j)$ is a solution of the dual equation (29) for the nonzero input $\tilde{u}(i,j)=u(N-i,M-j)$.

Proof. (a) Let $\tilde{x}(i,j)$ be a solution of (29) for the non-zero input $\tilde{u}(i,j)$. This implies that (29) is satisfied. Now consider equation (10). If we set $x(i,j)=\tilde{x}(N-i,M-j)$, $u(i,j)=\tilde{u}(N-i,M-j)$ we have

$$\begin{aligned}
&E_Ix(i+1,j)+E_{II}x(i,j+1)\\
&=E_I\tilde{x}(N-(i+1),M-j)+E_{II}\tilde{x}(N-i,M-(j+1))\\
&\overset{(29)}{=}A\tilde{x}(N-i,M-j)+B\tilde{u}(N-i,M-j)\\
&\underset{u(i,j)=\tilde{u}(N-i,M-j)}{\overset{(x(i,j)=\tilde{x}(N-i,M-j))}{=}}Ax(i,j)+Bu(i,j)
\end{aligned}$$

(b) In the same way we can prove the second part of the Theorem.

A direct result of Lemma 3 is that the backward solution of the singular Roesser model (10) comes directly from the forward solution of the dual singular Roesser model (29). In the next three subsections we give the forward, backward and symmetric solution of the singular Roesser model (10) in terms of the matrix coefficients E_I, E_{II}, A, B and the forward/backward fundamental matrix sequence $\{T_{i,j}\}/\{V_{i,j}\}$ of $G(z_1,z_2)^{-1}$.

4.1. The forward solution of the implicit Roesser model.

Consider the singular Roesser model (10) and the Laurent matrix expansion at infinity of $G(z_1,z_2)^{-1}$ given in (15). Then the unique forward solution to (10) with admissible (11) is given according to [12] by:

$$\begin{aligned}
x(i,j)&=\sum_{p=0}^{i+n_1}\sum_{q=0}^{j+n_2}T_{i-p,j-q}Bu(p,q)\\
&+\sum_{q=0}^{i+n_2}T_{i+1,j-q}E_Ix(0,q)+\sum_{p=0}^{i+n_1}T_{i-p,j+1}E_{II}x(p,0)
\end{aligned}\tag{30}$$

for $(-n_1,-n_2)\le(i,j)$. It is important to note that (10) does not always have a solution. A necessary and sufficient condition for (10) to have a solution is that the initial conditions (11) satisfy the relation (30) for ($i=0\&j=0,1,2,...,M$) and ($j=0\&i=0,1,...,N$).

4.2. The backward solution of the implicit Roesser model.

Let $F(z_1,z_2)$ be the $2-D$ Z-transform of a function $f(i,j)$ satisfying the condition $f(i,j)=0$ for $i<0$ or/and $j<0$ defined by [25]

$$F(z_1,z_2)=Z[f(i,j)]:=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}f(i,j)z_1^{-i}z_2^{-j}.\tag{31}$$

Lemma 4. [25] If $F(z_1,z_2)=Z[f(i,j)]$, then

$$Z[f(i+1,j)]=z_1[F(z_1,z_2)-F(0,z_2)]\tag{32}$$

$$Z[f(i,j+1)]=z_2[F(z_1,z_2)-F(z_1,0)]\tag{33}$$

$$Z\left[f(i+1,j+1)\right]= z_1z_2\{F(z_1,z_2)-F(z_1,0)-F(0,z_2)+f(0,0)\} \tag{34}$$

where $F(z_1,0) = \sum_{i=0}^{\infty} f(i,0)\,z_1^{-i}$, $F(0,z_2) = \sum_{j=0}^{\infty} f(0,j)\,z_2^{-j}$.

Consider the dual Roesser model of (10). Let also $\tilde{X}(z_1,z_2)=Z[\tilde{x}(i,j)]$ and $\tilde{U}(z_1,z_2)=Z[\tilde{u}(i,j)]$. Using (32), (33) and (34) for (29) we obtain

$$\begin{aligned} &E_I\tilde{x}(i,j+1)+E_{II}\tilde{x}(i+1,j)\\ &=A\tilde{x}(i+1,j+1)+B\tilde{u}(i+1,j+1)\overset{Z[\bullet]}{\Longrightarrow}\\ &E_Iz_2\left[\tilde{X}(z_1,z_2)-\tilde{X}(z_1,0)\right]\\ &+E_{II}z_1\left[\tilde{X}(z_1,z_2)-\tilde{X}(0,z_2)\right]=\\ &=A\left\{z_1z_2\left[\tilde{X}(z_1,z_2)-\tilde{X}(z_1,0)-\tilde{X}(0,z_2)+\tilde{x}(0,0)\right]\right\}+\\ &+B\left\{z_1z_2\left[\tilde{U}(z_1,z_2)-\tilde{U}(z_1,0)-\tilde{U}(0,z_2)+\tilde{u}(0,0)\right]\right\}\end{aligned}$$

or equivalently

$$\begin{aligned}\tilde{X}(z_1,z_2)=&\underbrace{\left(\sum_{p=-f_1}^{\infty}\sum_{q=-f_2}^{\infty}\tilde{T}_{p,q}z_1^{-p}z_2^{-q}\right)}_{(E_Iz_2+E_{II}z_1-Az_1z_2)^{-1}}\times\\ &\times\{Bz_1z_2\tilde{U}(z_1,z_2)-Bz_1z_2\tilde{U}(z_1,0)-Bz_1z_2\tilde{U}(0,z_2)\\ &+Bz_1z_2\tilde{u}(0,0)+Az_1z_2\tilde{x}(0,0)+\\ &+E_Iz_2\tilde{X}(z_1,0)-Az_1z_2\tilde{X}(z_1,0)+E_{II}z_1\tilde{X}(0,z_2)\\ &-Az_1z_2\tilde{X}(0,z_2)\}.\end{aligned} \tag{35}$$

Using the inverse 2-D transformation [25] for (35) and taking into account that $\tilde{T}_{p,q}=0$ for $p<-f_1$ or $q<-f_2$, we obtain

$$\begin{aligned}\tilde{x}(i,j)=&\sum_{p=0}^{i+f_1+1}\sum_{q=0}^{j+f_2+1}\tilde{T}_{i-p+1,j-q+1}B\tilde{u}(p,q)\\ &+\sum_{p=0}^{i+f_1}\tilde{T}_{i-p,j+1}E_I\tilde{x}(p,0)+\sum_{q=0}^{j+f_2}\tilde{T}_{i+1,j-q}E_{II}\tilde{x}(0,q)\\ &-\sum_{p=1}^{i+f_1+1}\tilde{T}_{i-p+1,j+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}\tilde{x}(p,0)\\ \tilde{u}(p,0)\end{bmatrix}\\ &-\sum_{q=1}^{j+f_2+1}\tilde{T}_{i+1,j-q+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}\tilde{x}(0,q)\\ \tilde{u}(0,q)\end{bmatrix}\\ &+\tilde{T}_{i+1,j+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}\tilde{x}(0,0)\\ \tilde{u}(0,0)\end{bmatrix}.\end{aligned} \tag{36}$$

Now by using the part (a) of Lemma 3 and the solution of the dual Roesser model (36) we can easily prove the following Theorem.

Theorem 5. If $\det[G(z_1,z_2)]\neq 0$, and the condition of Theorem 1 is satisfied, then the unique backward solution to (10) with admissible boundary conditions (12) is given by

$$\begin{aligned}x(i,j)=&\sum_{p=0}^{N-i+\ell_1}\sum_{q=0}^{M-j+\ell_2}V_{p-i-N,q-j-M}Bu(N-p,M-q)\\ &+\sum_{p=0}^{N-i+\ell_1-1}V_{1+p+i-N,j-M}E_Ix(N-p,M)\\ &+\sum_{q=0}^{M-j+\ell_2-1}V_{i-N,1+q+j-M}E_{II}x(N,M-q)\\ &-\sum_{p=1}^{N-i+\ell_1}V_{p+i-N,j-M}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N-p,M)\\ u(N-p,M)\end{bmatrix}\\ &-\sum_{q=1}^{M-j+\ell_2}V_{i-N,q+j-M}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N,M-q)\\ u(N,M-q)\end{bmatrix}\\ &+V_{i-N,j-M}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N,M)\\ u(N,M)\end{bmatrix}\end{aligned} \tag{37}$$

where $V_{i,j}$ is the backward fundamental matrix sequence of $G(z_1,z_2)^{-1}$ given in (16).

Proof. Let $\tilde{x}(i,j)$ be the solution of (29) for the non-zero input $\tilde{u}(i,j)$ presented in (36). Then the sequence $x(i,j)=\tilde{x}(N-i,M-j)$ is a solution of the dual Eq. (10) for the nonzero input $u(i,j)=\tilde{u}(N-i,M-j)$ or otherwise

$$\begin{aligned}x(i,j)&=\tilde{x}(N-i,M-j)\\ =&\sum_{p=0}^{N-i+f_1+1}\sum_{q=0}^{M-j+f_2+1}\tilde{T}_{N-i-p+1,M-j-q+1}Bu(N-p,M-q)\\ &+\sum_{p=0}^{N-i+f_1}\tilde{T}_{N-i-p,M-j+1}E_Ix(N-p,M)\\ &+\sum_{q=0}^{M-j+f_2}\tilde{T}_{N-i+1,M-j-q}E_{II}x(N,M-q)\\ &-\sum_{p=1}^{N-i+f_1+1}\tilde{T}_{N-i-p+1,M-j+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N-p,M)\\ u(N-p,M)\end{bmatrix}\\ &-\sum_{q=1}^{M-j+f_2+1}\tilde{T}_{N-i+1,M-j-q+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N,M-q)\\ u(N,M-q)\end{bmatrix}\\ &+\tilde{T}_{N-i+1,M-j+1}\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x(N,M)\\ u(N,M)\end{bmatrix}\end{aligned}$$

or by using (27) we have (37).

A necessary and sufficient condition for (10) to have a solution is that the final conditions (12) satisfy (37) for ($i=N\&j=0,1,...,M$) and ($i=0,1,....,N\&j=M$).

4.3. The symmetric solution of the implicit Roesser model. Consider the Laurent expansion at infinity of $G^{-1}(z_1, z_2)$ given in (15). Then the following relations

$$-T_{p-1,q-1}A + T_{p,q-1}E_I + T_{p-1,q}E_{II} = \delta_{p-1,q-1}I_n$$

follow from comparison of coefficient matrices at like powers of z_1 and z_2 of the equality

$$\underbrace{\left(\sum_{p=-n_1}^{\infty}\sum_{q=-n_2}^{\infty} T_{p,q}z_1^{-p}z_2^{-q}\right)}_{G(z_1,z_2)^{-1}} \times \underbrace{(E_I z_1 + E_{II} z_2 - A)}_{G(z_1,z_2)} = I_n.$$

Define now the matrices

$$\mathcal{A}_0 = \begin{pmatrix} E_1 & -A & \cdots & 0 & 0 \\ 0 & E_I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A & 0 \\ 0 & 0 & \cdots & E_I & -A \end{pmatrix} \in R^{nN \times n(N+1)}$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & E_{II} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_{II} & 0 \\ 0 & 0 & \cdots & 0 & E_{II} \end{pmatrix} \in R^{nN \times n(N+1)}$$

$$\mathcal{B} = \begin{pmatrix} 0 & B & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 \\ 0 & 0 & \cdots & 0 & B \end{pmatrix} \in R^{nN \times m(N+1)}$$

and the vectors

$$y_i = \begin{pmatrix} x_{N,i} \\ x_{N-1,i} \\ \vdots \\ x_{1,i} \\ x_{0,i} \end{pmatrix} \in R^{(N+1)n},$$

$$u_i = \begin{pmatrix} u_{N,i} \\ u_{N-1,i} \\ \vdots \\ u_{1,i} \\ u_{0,i} \end{pmatrix} \in R^{(N+1)m}$$

$$i = 0, 1, ..., M.$$

Then (10) may be rewritten in the form

$$\underbrace{\begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_0 & \cdots & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_1 & \mathcal{A}_0 & 0 \\ 0 & 0 & \cdots & 0 & \mathcal{A}_1 & \mathcal{A}_0 \end{pmatrix}}_{\tilde{A}_N} \underbrace{\begin{pmatrix} y_M \\ y_{M-1} \\ \vdots \\ y_1 \\ y_0 \end{pmatrix}}_{y_{0,M}}$$
$$= \underbrace{\begin{pmatrix} 0 & \mathcal{B} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{B} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathcal{B} \end{pmatrix}}_{\tilde{B}_N} \underbrace{\begin{pmatrix} u_M \\ u_{M-1} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix}}_{v_{0,M}} \tag{38}$$

Let also

$$\mathcal{H}_i = \begin{pmatrix} T_{1,i} & T_{2,i} & \cdots & T_{N-1,i} & T_{N,i} \\ T_{0,i} & T_{1,i} & \cdots & T_{N-2,i} & T_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{-N+1,i} & T_{-N+2,i} & \cdots & T_{0,i} & T_{1,i} \\ T_{-N,i} & T_{-N+1,i} & \cdots & T_{-1,i} & T_{0,i} \end{pmatrix}$$

Then we can check that

$$\underbrace{\begin{pmatrix} T_{1,i} & T_{2,i} & \cdots & T_{N-1,i} & T_{N,i} \\ T_{0,i} & T_{1,i} & \cdots & T_{N-2,i} & T_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{-N+2,i} & T_{-N+3,i} & \cdots & T_{1,i} & T_{2,i} \\ T_{-N+1,i} & T_{-N+2,i} & \cdots & T_{0,i} & T_{1,i} \end{pmatrix}}_{\mathcal{H}_i} \underbrace{\begin{pmatrix} 0 & E_{II} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_{II} & 0 \\ 0 & 0 & \cdots & 0 & E_{II} \end{pmatrix}}_{\mathcal{A}_1} +$$
$$\underbrace{\begin{pmatrix} T_{1,i-1} & T_{2,i-1} & \cdots & T_{N-1,i-1} & T_{N,i-1} \\ T_{0,i-1} & T_{1,i-1} & \cdots & T_{N-2,i-1} & T_{N-1,i-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{-N+2,i-1} & T_{-N+3,i-1} & \cdots & T_{1,i-1} & T_{2,i-1} \\ T_{-N+1,i-1} & T_{-N+2,i-1} & \cdots & T_{0,i-1} & T_{1,i-1} \end{pmatrix}}_{\mathcal{H}_{i-1}} \underbrace{\begin{pmatrix} E_I & -A & \cdots & 0 & 0 \\ 0 & E_I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A & 0 \\ 0 & 0 & \cdots & E_I & -A \end{pmatrix}}_{\mathcal{A}_0} = \mathcal{S}_i$$

where

$$\mathcal{S}_i = \begin{pmatrix} F_{1,i} & 0 & \cdots & 0 & Q_{N,i} \\ F_{0,i} & \delta_{i-1}I & \cdots & 0 & Q_{N-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{-N+2,i} & 0 & \cdots & \delta_{i-1}I & Q_{2,i} \\ F_{-N+1,i} & 0 & \cdots & 0 & Q_{1,i} \end{pmatrix}$$

$$F_{k,i} = T_{k,i-1}E_I$$
$$Q_{k,i} = T_{k,i}E_{II} - T_{k,i-1}A.$$

Premultiplying (38) by the matrix

$$\tilde{A}_N^L = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots & \mathcal{H}_M \\ \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \cdots & \mathcal{H}_{M-1} \\ \mathcal{H}_{-1} & \mathcal{H}_0 & \mathcal{H}_1 & \cdots & \mathcal{H}_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{-M+1} & \mathcal{H}_{-M+2} & \mathcal{H}_{-M+3} & \cdots & \mathcal{H}_0 \end{pmatrix}$$

we obtain that

$$\tilde{A}_N^L \tilde{A}_N y_{0,M} = \tilde{A}_N^L \tilde{B}_N v_{0,M} \Leftrightarrow$$

$$\begin{pmatrix} \mathcal{H}_1\mathcal{A}_1 & \mathcal{S}_2 & \cdots & \mathcal{S}_M & \mathcal{H}_M\mathcal{A}_0 \\ \mathcal{H}_0\mathcal{A}_1 & \mathcal{S}_1 & \cdots & \mathcal{S}_{M-1} & \mathcal{H}_{M-1}\mathcal{A}_0 \\ \mathcal{H}_{-1}\mathcal{A}_1 & \mathcal{S}_0 & \cdots & \mathcal{S}_{M-2} & \mathcal{H}_{M-2}\mathcal{A}_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{H}_{-M+1}\mathcal{A}_1 & \mathcal{S}_{-M+2} & \cdots & \mathcal{S}_0 & \mathcal{H}_0\mathcal{A}_0 \end{pmatrix} \underbrace{\begin{pmatrix} y_M \\ y_{M-1} \\ \vdots \\ y_1 \\ y_0 \end{pmatrix}}_{y_{0,M}} = \tag{39}$$

$$= \begin{pmatrix} 0 & \mathcal{H}_1\mathcal{B} & \cdots & \mathcal{H}_M\mathcal{B} \\ 0 & \mathcal{H}_0\mathcal{B} & \cdots & \mathcal{H}_{M-1}\mathcal{B} \\ 0 & \mathcal{H}_{-1}\mathcal{B} & \cdots & \mathcal{H}_{M-2}\mathcal{B} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathcal{H}_{-M+1}\mathcal{B} & \cdots & \mathcal{H}_0\mathcal{B} \end{pmatrix} \underbrace{\begin{pmatrix} u_M \\ u_{M-1} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix}}_{v_{0,M}}.$$

From the first and last block equation we get boundary conditions that must be satisfied in order for (10) to have a solution:

$$\begin{aligned} &\mathcal{H}_1\mathcal{A}_1 y_M + \mathcal{S}_2 y_{M-1} + \cdots + \mathcal{S}_M y_1 + \mathcal{H}_M\mathcal{A}_0 y_0 \\ &= (\mathcal{H}_1\mathcal{B})\, u_{M-1} + \cdots + (\mathcal{H}_{M-1}\mathcal{B})\, u_1 + (\mathcal{H}_M\mathcal{B})\, u_0 \end{aligned} \tag{40}$$

and

$$\begin{aligned} &\mathcal{H}_{-M+1}\mathcal{A}_1 y_M + \mathcal{S}_{-M+2} y_{M-1} + \cdots + \mathcal{S}_0 y_1 + \mathcal{H}_0\mathcal{A}_0 y_0 \\ &= (\mathcal{H}_{-M+1}\mathcal{B})\, u_{M-1} + \cdots + (\mathcal{H}_0\mathcal{B})\, u_1 + (\mathcal{H}_0\mathcal{B})\, u_0. \end{aligned} \tag{41}$$

Note that the matrices $S_i, i = 0, 2, 3, ..., M$ in (40) and (41) have all their block columns, except of the first and the last one, filled with zero entries and therefore the above equations gives rise only to boundary conditions of the form (13). Now consider the remaining equations that arise from (39)

$$\begin{aligned} &(\mathcal{H}_{-q}\mathcal{A}_1)\, y_M + \mathcal{S}_{-q+1} y_{M-1} + \cdots + \mathcal{S}_{-q+M-1} y_1 \\ &+ (\mathcal{H}_{-q+M-1}\mathcal{A}_0)\, y_0 = (\mathcal{H}_{-q}\mathcal{B})\, u_{M-1} + \cdots \\ &+ (\mathcal{H}_{-q+M-2}\mathcal{B})\, u_1 + (\mathcal{H}_{-q+M-1}\mathcal{B})\, u_0 \end{aligned}$$

where $q = 0, 1, ..., M-2$, or equivalently

$$\begin{pmatrix} T_{1,-q} & T_{2,-q} & \cdots & T_{N,-q} \\ T_{0,-q} & T_{1,-q} & \cdots & T_{N-1,-q} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-N+2,-q} & T_{-N+3,-q} & \cdots & T_{2,-q} \\ T_{-N+1,-q} & T_{-N+2,-q} & \cdots & T_{1,-q} \end{pmatrix} \begin{pmatrix} 0 & E_{II} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_{II} & 0 \\ 0 & 0 & \cdots & 0 & E_{II} \end{pmatrix} \begin{pmatrix} x_{N,M} \\ x_{N-1,M} \\ \vdots \\ x_{1,M} \\ x_{0,M} \end{pmatrix}$$

$$+ \sum_{j=-q+1}^{-q+M-1} \begin{pmatrix} F_{1,j} & 0 & \cdots & 0 & Q_{N,j} \\ F_{0,j} & \delta_{i-1}I & \cdots & 0 & Q_{N-1,j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{-N+2,j} & 0 & \cdots & \delta_{i-1}I & Q_{2,j} \\ F_{-N+1,j} & 0 & \cdots & 0 & Q_{1,j} \end{pmatrix} \begin{pmatrix} x_{N,M-1-j} \\ x_{N-1,M-1-j} \\ \vdots \\ x_{1,M-1-j} \\ x_{0,M-1-j} \end{pmatrix}$$

$$+ \begin{pmatrix} T_{1,-q+M-1} & T_{2,-q+M-1} & \cdots & T_{N,-q+M-1} \\ T_{0,-q+M-1} & T_{1,-q+M-1} & \cdots & T_{N-1,-q+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-N+2,-q+M-1} & T_{-N+3,-q+M-1} & \cdots & T_{2,-q+M-1} \\ T_{-N+1,-q+M-1} & T_{-N+2,-q+M-1} & \cdots & T_{1,-q+M-1} \end{pmatrix} \begin{pmatrix} E_I & -A & \cdots & 0 & 0 \\ 0 & E_I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A & 0 \\ 0 & 0 & \cdots & E_I & -A \end{pmatrix} \begin{pmatrix} x_{N,0} \\ x_{N-1,0} \\ \vdots \\ x_{1,0} \\ x_{0,0} \end{pmatrix}$$

$$= \sum_{j=-q}^{-q+M-2} \{ \begin{pmatrix} T_{1,j} & T_{2,j} & \cdots & T_{N,j} \\ T_{0,j} & T_{1,j} & \cdots & T_{N-1,j} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-N+2,j} & T_{-N+3,j} & \cdots & T_{2,j} \\ T_{-N+1,j} & T_{-N+2,j} & \cdots & T_{1,j} \end{pmatrix}$$

$$
\begin{pmatrix} 0 & B & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 \\ 0 & 0 & \cdots & 0 & B \end{pmatrix}
$$

$$
+\begin{pmatrix} T_{1,-q+M-1} & T_{2,-q+M-1} & \cdots & T_{N,-q+M-1} \\ T_{0,-q+M-1} & T_{1,-q+M-1} & \cdots & T_{N-1,-q+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-N+2,-q+M-1} & T_{-N+3,-q+M-1} & \cdots & T_{2,-q+M-1} \\ T_{-N+1,-q+M-1} & T_{-N+2,-q+M-1} & \cdots & T_{1,-q+M-1} \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & B & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 \\ 0 & 0 & \cdots & 0 & B \end{pmatrix}\begin{pmatrix} u_{N,0} \\ u_{N-1,0} \\ \vdots \\ u_{1,0} \\ u_{0,0} \end{pmatrix} \tag{42}
$$

or equivalently by taking the $i-$th row of the above equations i.e. for $q=0,1,...,M-2$ and $i=0,1,...,N-2$

$$
\begin{gathered}
T_{-i+N-1,-q}E_{II}x_{0,M}+\sum_{k=0}^{N-2}\left(T_{-i+k,-q}E_{II}\right)x_{N-1-k,M} \\
+\sum_{j=1-q}^{M-q}\{\left(T_{-i,j-1}E_I\right)x_{N,M-q+1-j} \\
+\left(T_{-i+N-1,j}E_{II}-T_{-i+N-1,j-1}A\right)x_{0,M-q+1-j}\} \\
+x_{N-1+i,M-1-q}+T_{-i,M-1-q}E_Ix_{N,0} \\
-\sum_{k=0}^{N-2}\left(T_{-i+k,M-1-q}A-T_{-i+k+1,M-1-q}E_I\right)x_{N-1-k,0} \\
-T_{-i+N-1,M-1-q}Ax_{0,0}= \\
=\sum_{j=-q}^{M-2-q}\sum_{k=0}^{N-2}\{T_{-i+k,j}B\}u_{N-k-1,M-1-j} \\
+\left(T_{-i+N-1,j}B\right)u_{0,M-1-j}\}+\left(T_{-i+N-1,M-1-q}B\right)u_{0,0} \\
+\sum_{k=0}^{N-2}\left(T_{-i+k,M-1-q}B\right)u_{N-1-k,0}.
\end{gathered}
$$

Now by substituting $N-1+i$ with p, and $M-1-q$ with q, we can easily get the following Theorem.

Theorem 6. If $\det\left[G\left(z_1,z_2\right)\right]\neq 0$, and $\deg_z\left|G\left(z,z\right)\right|=\deg_{z_1}\left|G\left(z_1,z_2\right)\right|+\deg_{z_2}\left|G\left(z_1,z_2\right)\right|$ is satisfied [12], then the unique symmetric solution to (10) with admissible boundary conditions (13) is given by

$$
\begin{gathered}
x_{p,q}=-T_{2(N-1)-p,1+q-M}E_{II}x_{0,M} \\
-\sum_{k=0}^{N-2}T_{N-1-p+k,1+q-M}E_{II}x_{N-1-k,M} \\
+\sum_{j=q+2-M}^{q+1}\{-T_{N-1-p,j-1}E_Ix_{N,q+2-j} \\
+\{T_{2(N-1)-p,j}E_{II}-T_{2(N-1)-p,j-1}A\}x_{0,q+2-j}\} \\
-T_{N-1-p,q}E_Ix_{N,0}+\sum_{k=0}^{N-2}\{T_{N-1-p+k,q}A-T_{N-p+k,q}E_I\} \\
x_{N-1-k,0}+T_{2(N-1)-p,q}Ax_{0,0} \\
+\sum_{j=q+1-M}^{q-1}\sum_{k=0}^{N-2}T_{N-1-p+k,j}Bu_{N-k-1,M-1-j} \\
+\sum_{j=q+1-M}^{q-1}T_{2(N-1)-p,j}Bu_{0,M-1-j}+T_{2(N-1)-p,q}Bu_{0,0} \\
+\sum_{k=0}^{N-2}\{T_{N-1+p+k,q}B\}u_{N-1-k,0}.
\end{gathered}
$$

Using now the first and last block row equations of (42) we get the following extra boundary conditions for ($i=-1,N-1$ & $q=0,1,...,M-2$), or ($q=-1,M-1$ & $i=-1,0,...,N-2,N-1$) (the boundary equations that we have described before in terms of block matrices)

$$
\begin{gathered}
T_{-i+N-1,-q}E_{II}x_{0,M}+\sum_{k=0}^{N-2}T_{-i+k,-q}E_{II}x_{N-1-k,M} \\
+\sum_{j=1-q}^{M-q}\{T_{-i,j-1}E_Ix_{N,M-q+1-j} \\
+\left(T_{-i+N-1,j}E_{II}-T_{-i+N-1,j-1}A\right)x_{0,M-q+1-j}\} \\
-T_{-i,M-1-q}E_Ix_{N,0}-T_{-i+N-1,M-1-q}Ax_{0,0} \\
-\sum_{k=0}^{N-2}\left\{T_{-i+k,M-1-q}A-T_{-i+k+1,M-1-q}E_I\right\}x_{N-1-k,0} \\
=\sum_{j=-q}^{M-2-q}\sum_{k=0}^{N-2}T_{-i+k,j}Bu_{N-k-1,M-1-j} \\
+\sum_{j=-q}^{M-2-q}T_{-i+N-1,j}Bu_{0,M-1-j}+\left(T_{-i+N-1,M-1-q}B\right)u_{0,0} \\
+\sum_{k=0}^{N-2}T_{-i+k,M-1-q}Bu_{N-1-k,0}.
\end{gathered} \tag{43}
$$

Therefore, a necessary and sufficient condition so that (10) has a solution is that the initial conditions, final conditions and input sequences satisfy the relations (13), (40), (41) and (43).

5. Example

Consider a nonrecursible mask described by the difference equation [12]

$$y_{i,j} = y_{i-1,j-1} + y_{i-1,j+1} + y_{i+1,j-1} + y_{i+1,j+1} + u_{i,j}$$

where $u_{i,j}$ the input and $y_{i,j}$ the output

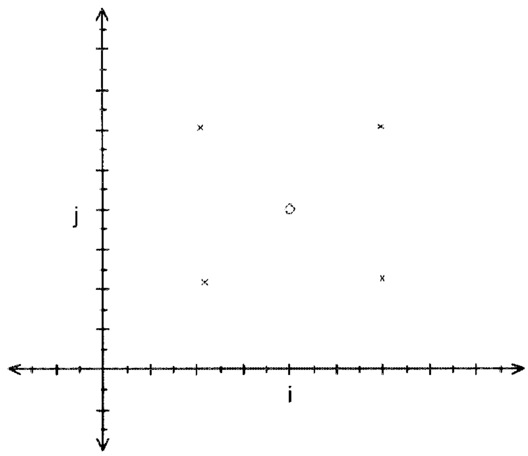

Fig. 1. Nonrecursible mask

A singular realization according to [12] of the above nonrecusrible mask is the following:

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} x^{h_1}_{i+1,j} \\ x^{h_2}_{i+1,j} \\ x^{h_3}_{i+1,j} \\ \hline x^{v_1}_{i,j+1} \\ x^{v_2}_{i,j+1} \\ x^{v_3}_{i,j+1} \end{array}\right]}_{E_I x_{i+1,j} + E_{II} x_{i,j+1}}$$

$$= \underbrace{\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]}_{A}\underbrace{\left[\begin{array}{c} x^{h_1}_{i,j} \\ x^{h_2}_{i,j} \\ x^{h_3}_{i,j} \\ \hline x^{v_1}_{i,j} \\ x^{v_2}_{i,j} \\ x^{v_3}_{i,j} \end{array}\right]}_{x_{i,j}} + \underbrace{\left[\begin{array}{c} 1 \\ 0 \\ -1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]}_{B} u_{i,j}$$

$$y_{i,j} = \underbrace{\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]}_{C} x_{i,j} + u_{i,j}$$

where

$$x_{i,j} = \left[\begin{array}{c} x^h_{i,j} \\ x^v_{i,j} \end{array}\right] = \left[\begin{array}{c} x^{h_1}_{i,j} \\ x^{h_2}_{i,j} \\ x^{h_3}_{i,j} \\ \hline x^{v_1}_{i,j} \\ x^{v_2}_{i,j} \\ x^{v_3}_{i,j} \end{array}\right]$$

$$E_I = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$E_{II} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Note that nonrecursible masks cannot be represented using state-space Roesser models. Define the 2-D matrix pencil

$$G(z_1, z_2) = z_1 E_I + z_2 E_{II} - A$$

$$= \left[\begin{array}{ccc|ccc} z_1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & z_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ \hline -1 & -1 & 0 & z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & z_2 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right].$$

Using the algorithm presented in [12], we have that

$$\det G(z_1, z_2) = \sum_{i=0}^{2}\sum_{j=0}^{2} d_{i,j} z_1^i z_2^j = -z_1^2 z_2^2 - z_1^2 + z_1 z_2 - z_2^2 - 1$$

$$R(z_1, z_2) = \sum_{i=0}^{2}\sum_{j=0}^{2} R_{i,j} z_1^i z_2^j =$$

$$= \left(\begin{array}{ccc} -z_1 z_2^2 + z_2 - z_1 & -z_2^2 - 1 & -z_1 z_2^2 - z_1 \\ z_1 z_2^2 + z_1 & z_2^2 - z_1 z_2 + 1 & -z_2 z_1^2 + z_2^2 z_1 + z_1 \\ z_2^2 + 1 & -z_1 z_2^2 - z_1 & z_2^2 - z_1 z_2 + 1 \\ 1 & -z_1 & -z_1^2 \\ z_2^2 & -z_1 z_2^2 & -z_1^2 z_2^2 \\ z_2 & -z_1 z_2 & -z_1^2 z_2 \end{array}\right.$$

$$\left.\begin{array}{ccc} 1 & -z_2 & -z_2^2 \\ z_1^2 & -z_1^2 z_2 & -z_1^2 z_2^2 \\ z_1 & -z_1 z_2 & -z_1 z_2^2 \\ -z_2 z_1^2 + z_1 - z_2 & -z_1^2 - 1 & -z_2 z_1^2 - z_2 \\ z_2 z_1^2 + z_2 & z_1^2 - z_1 z_2 + 1 & z_1^2 z_2 - z_1 z_2^2 + z_2 \\ z_1^2 + 1 & -z_1^2 z_2 - z_2 & z_1^2 - z_1 z_2 + 1 \end{array}\right)$$

and since $4 = \deg_z |G(z,z)| = \deg_{z_1} |G(z_1, z_2)| + \deg_{z_2} |G(z_1, z_2)| = 2 + 2$ is satisfied therefore the *Laurent expansion at infinity* of $G(z_1, z_2)^{-1}$ exists, and is unique. Similarly since $0 = d^d = d_1^d + d_2^d = 0 + 0$, where d^d is the least degree in z of $\det G(z,z) = \det(zE_I + zE_{II} - A)$ and d_i^d are the least degrees of $d(z_1, z_2)$ in z_i, or equivalently that $-1 = d_{0,0} = d_{d_1^d, d_2^d} \neq 0$, the Laurent expansion at zero of $G(z_1, z_2)^{-1}$ is unique. We have also that

$$D_0 = P_0^{-1} = \begin{pmatrix} d_{0,0}I_6 & 0 & 0 \\ d_{1,0}I_6 & d_{0,0}I_6 & 0 \\ d_{2,0}I_6 & d_{1,0}I_6 & d_{0,0}I_6 \end{pmatrix}^{-1} = \begin{pmatrix} r_0 I_6 & 0 & 0 \\ r_1 I_6 & r_0 I_6 & 0 \\ r_2 I_6 & r_1 I_6 & r_0 I_6 \end{pmatrix}$$

where

$$\begin{aligned} r_0 &= \frac{1}{d_{0,0}} = -1 \\ r_1 &= -\frac{1}{d_{0,0}}\left[d_{1,0} \times r_0\right] = 0 \\ r_2 &= -\frac{1}{d_{0,0}}\left[d_{2,0} \times r_0 + d_{1,0} \times r_1\right] = 1. \end{aligned}$$

Then by setting

$$R_0 = \begin{pmatrix} R_{0,0} \\ R_{1,0} \\ R_{2,0} \end{pmatrix}; \; R_1 = \begin{pmatrix} R_{0,1} \\ R_{1,1} \\ R_{2,1} \end{pmatrix}; \; R_2 = \begin{pmatrix} R_{0,2} \\ R_{1,2} \\ R_{2,2} \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} d_{0,1}I_6 & 0 & 0 \\ d_{1,1}I_6 & d_{0,1}I_6 & 0 \\ d_{2,1}I_6 & d_{1,1}I_6 & d_{0,1}I_6 \end{pmatrix};$$

$$P_2 = \begin{pmatrix} d_{0,2}I_6 & 0 & 0 \\ d_{1,2}I_6 & d_{0,2}I_6 & 0 \\ d_{2,2}I_6 & d_{1,2}I_6 & d_{0,2}I_6 \end{pmatrix}$$

we get

$$\begin{aligned} V_0 &:= \begin{pmatrix} V_{0,0} \\ V_{-1,0} \\ V_{-2,0} \end{pmatrix} = D_0 R_0 \\ V_{-1} &:= \begin{pmatrix} V_{0,-1} \\ V_{-1,-1} \\ V_{-2,-1} \end{pmatrix} = D_0 R_1 + D_1 R_0 \\ V_{-2} &:= \begin{pmatrix} V_{0,-2} \\ V_{-1,-2} \\ V_{-2,-2} \end{pmatrix} = D_0 R_2 + D_1 R_1 + D_2 R_0 \end{aligned}$$

and thus

$$V_{0,0} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix};$$

$$V_{-1,0} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$V_{-2,0} = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$V_{0,-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

....

The rest terms of the Laurent expansion at zero of $G(z_1, z_2)^{-1}$ are given by

$$\begin{aligned} V_{-i,-j} &= -\frac{1}{d_{0,0}} \sum_{l=0}^{2} \sum_{m=0}^{2} d_{l,m} V_{-i+l,-j+m} \quad (l,m) \neq (0,0) \\ &= d_{2,0} V_{-i+2,-j} + d_{0,2} V_{-i,-j+2} + d_{1,1} V_{-i+1,-j+1} \\ &+ d_{2,2} V_{-i+2,-j+2} = \\ &= -V_{-i+2,-j} - V_{-i,-j+2} + V_{-i+1,-j+1} \\ &- V_{-i+2,-j+2} \underset{j \leftarrow -j}{\overset{i \leftarrow -i}{\Longleftrightarrow}} \\ V_{i,j} &= -V_{i+2,j} - V_{i,j+2} + V_{i+1,j+1} - V_{i+2,j+2}. \end{aligned}$$

For example

$$V_{-3,-2} = -V_{-1,-2} - V_{-3,0} + V_{-2,-1} - V_{-1,0} =$$
$$= -V_{-1,-2} - (-V_{-1,0} - V_{-3,2} + V_{-2,-2} - V_{-1,2})$$
$$+V_{-2,-1} - V_{-1,0} =$$
$$= -V_{-1,-2} - (-V_{-1,0} + V_{-2,-2}) + V_{-2,-1} - V_{-1,0} =$$
$$= -V_{-1,-2} + V_{-2,-2} + V_{-2,-1} =$$
$$= \begin{pmatrix} 3 & 0 & 1 & 0 & -2 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 2 & 0 & -4 & 0 & -2 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

Note that the backward fundemental matrix sequence may be used to compute $x(i,j)$ using (37). For instance, suppose that we are interested for the backward solution of the system in the interval $[0,5] \times [0,5]$ and we know the final conditions $x(5,i), x(i,5), i = 0,1,..,5$ and the input $u(i,j), i,j \in \{0,1,..,5\}$. Then

$$x(4,4) = V_{-9,-9}Bu(5,5) + V_{-9,-8}Bu(5,4)$$
$$+V_{-8,-9}Bu(4,5) + V_{-8,-8}Bu(4,4)$$
$$+V_{0,-1}E_I x(5,5) + V_{-1,0}E_{II}x(5,5) - V_{0,-1}Ax(4,5)$$
$$-V_{0,-1}Bu(4,5) - V_{-1,0}Ax(5,4)$$
$$-V_{0,-1}Bu(5,4) + V_{-1,-1}Ax(5,5) + V_{-1,-1}Bu(5,5) =$$
$$= \begin{pmatrix} 354u_{5,4} - 375u_{4,5} - 2x^{h_1}_{5,5} + x^{v_1}_{4,5} - x^{h_3}_{5,4} \\ -153u_{5,4} + 143u_{4,5} + x^{h_3}_{5,4} \\ -377u_{5,5} - 153u_{4,4} - 2x^{v_1}_{5,5} + x^{h_1}_{5,4} \\ -578u_{5,5} - 233u_{4,4} - 2x^{v_1}_{5,5} - x^{v_3}_{4,5} + x^{h_1}_{5,4} \\ 201u_{5,5} + 80u_{4,4} + x^{v_3}_{4,5} \\ 201u_{5,4} - 232u_{4,5} - 2x^{h_1}_{5,5} + x^{v_1}_{4,5} \end{pmatrix}.$$

A computer program for the computation of the fundamental matrix sequence and its use in the computation of the local semistate is extremely useful.

6. Conclusions

In the case of discrete time implicit Roesser models, exact solutions where proposed in two different forms: a) backward solutions, and b) symmetric solutions. All the closed formula solutions were represented in terms of the forward and backward fundamental matrix of the implicit Roesser model. It is easily seen that the proposed solutions: a) are extensions of the ones proposed in [13] for 1-D discrete time singular systems, and b) accomplish the work that have been done by [11] and [12] for the forward solution of the general singular model and the implicit Roesser model respectively. An algorithm has also been provided for the computation of the backward fundamental matrix sequence, that is useful in the implementation of the proposed closed solution formulae. The computation of the forward and backward fundamental matrix sequence of a non-causal system might be useful in the solution of the descriptor system realization problem as has been studied by [26]. Certain controllability and observability criteria based on the proposed solutions are being studied and will be discussed in a future work.

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