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Modelling of non-linear long water waves on a sloping beach

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Abstract. The paper deals with a non-linear problem of long water waves approaching a sloping beach. In order to describe the phenomenon we apply the Lagrange's system of material variables. With these variables it is much easier to solve boundary conditions, especially conditions on a shoreline. The formulation is based on the fundamental assumption for long waves propagating in shallow water of constant depth that vertical material lines of fluid particles remain vertical during entire motion of the fluid. The analysis is confined to one – dimensional case of unsteady water motion within a 'triangular' body of fluid. The partial differential equations of fluid motion, obtained by means of a variational procedure, are then substituted by a system of equations resulting from a perturbation scheme with the second order expansion with respect to a small parameter. In this way the original problem has been reduced to a system of linear partial differential equations with variable coefficients. The latter equations are, in turn, substituted by a system of difference equations, which are then integrated in a discrete time space by means of the Wilson-θ method. The procedure developed in this paper may be a convenient tool in analysing non-breaking waves propagating in coastal zones of seas. Moreover, the model can also deliver useful results for cases when breaking of waves near a shoreline may be expected.

Key words: shallow water, non-linear wave, unsteady motion, sloping beach.

1. Introduction

In the description of long water waves propagating in shallow water a vertical momentum equation is usually assumed in such a form that the equation may be integrated independently from the equations corresponding to horizontal variables. Particular forms of the vertical acceleration term in the vertical momentum equation lead to different forms of the shallow water equations. For instance, two different approximations in the description of pressure field have lead to two classical theories of long water waves. The first one, known as the Airy's theory of very long waves, is based on the assumption that the pressure is hydrostatic. In the second approach, developed independently by Boussinesq and Korteweg and de Vries, the fluid acceleration influences the pressure field [1].

The theory formulated by Korteweg and de Vries [2] corresponds to potential motion of an incompressible fluid. The assumption of potential motion of the fluid and description of the problem in space variables simplifies the analysis. A drawback of such a formulation in the space variables is a solution to boundary conditions, especially on moving boundaries of the fluid domain. A solution to the initial value problem for long waves of small amplitude approaching a sloping beach belongs to Carrier and Greenspan [3]. The authors discovered a transformation allowing them to reduce non-linear shallow water equations to a linear differential equation for the fluid velocity. With the equation, they calculated the run-up height of non-breaking long waves on a sloping beach. Shuto [4] analysed a similar problem described in material variables. Like in the Eulerian description, a vertical acceleration term does not appear in a first order approximation of the momentum equations. Theoretical results obtained were compared with experimental data. A detailed discussion of the long waves phenomenon, together with a wide bibliography of the subject, may be found in Dingemans [5].

In the classical theories, the governing equations of fluid motion, written in the space variables, correspond to an average horizontal component of the velocity field over the fluid depth. It means that horizontal displacements of fluid particles forming a vertical material segment are equal to a common value depending on space variables and time. This kinematic assumption has been a starting point in a description of the long water waves phenomenon in material variables formulated by Wilde [6]. Like in the previous cases, in the latter approach the vertical material lines of fluid particles remain vertical during entire motion of the fluid. With the latter formulation it is much easier to solve boundary conditions. The price for it however is a more complicated structure of the equations of fluid motion.

In the present paper the problem of long water waves approaching a shoreline is considered. The discussion is limited to a plane problem, which, in our formulation, is transformed to one – dimensional in space, time dependent problem of non-breaking waves on a sloping beach. The analysis is performed with the help of material description of the phenomenon, and, in a sense, it is a generalisation of the problem discussed by Wilde [6]. Like in the latter approach, it has been assumed that vertical material lines, formed by the fluid particles, remain vertical during entire motion of the fluid. In our case, the analysis is confined to the description of motion of a triangular fluid domain with sloping bottom, starting to move at certain moment of time.

Fundamental equations of the model considered have been derived by means of a variational procedure with prescribed displacement field in the material co-ordinates. The resulting non-linear partial differential equations of the fluid motions are then substituted by a system of linear equations obtained

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by means of a perturbation scheme. As compared to the previous approaches, the formulation in the present paper seems to be less restrictive one.

2. Fundamental equations of an unsteady motion

In what follows we consider a plane problem of fluid motion in the Euclidean space. We confine our attention to the triangular fluid body shown schematically in Fig. 1. The motion of the fluid is induced by an assumed horizontal motion of the generator A-D starting to move at a certain moment of time. For the assumed harmonic motion of the generator, after a finite elapse of time from the starting point, the water surface waves will reach the shoreline. At the same time the run up of the shoreline occurs and thus, the material point C in the figure will be shifted along the slope. In order to describe the fluid motion we introduce the Cartesian system of co-ordinates in an actual configuration $(z^r,\ r=1,2)$, and a similar system in a reference configuration denoted by capital letters $(Z^\lambda,\ \lambda=1,2)$. The co-ordinates of the latter system define names of the fluid particles (positions of the particles at an initial moment of time). Moreover, it is convenient to introduce a common Cartesian system of co-ordinates. The motion of the fluid is described as the mapping of the names into actual positions occupied by the material points

$$\begin{aligned} z^1(Z^\alpha,t) &= Z^1 + u(Z^1,t), \\ z^2(Z^\alpha,t) &= Z^2 + v(Z^\alpha,t), \end{aligned} \tag{1}$$

in which $\alpha = 1,2$, and $u(Z^1,t)$ and $v(Z^\alpha,t)$ are components of the displacement vector of the fluid particle (Z^1,Z^2).

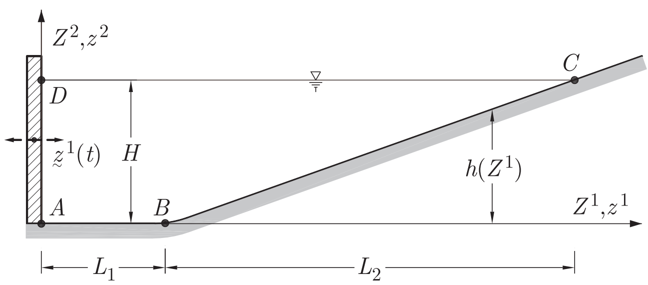

Fig. 1. Finite fluid domain with sloping bottom

With respect to the above, it is assumed the vertical component of the displacement field

$$v(Z^\alpha,t) = h(Z^1+u) - h(Z^1) + \frac{w(Z^1,t)}{H-h(Z^1)}\left[Z^2 - h(Z^1)\right]. \tag{2}$$

In the equation, the first two terms on the right hand side denote a rigid vertical displacement of the water column $Z^1 =$ const, and $w(Z^1,t)$ is the additional vertical component of displacement of a material point of the free surface. The fluid body vertical shift entering the equation may be expressed as

$$h(Z^1+u) - h(Z^1) = uh'(Z^1) + \frac{1}{2}u^2h''(Z^1) + \cdots, \tag{3}$$

where the primes denote differentiation with respect to Z^1 and $h'(Z^1)$ is the bottom slope.

In the discussed case, shown in Fig. 1, the segments AB and BC have different, but constant slopes. For small horizontal displacements it is justified to neglect higher order terms in expansion (3) and confine our attention to the linear term in the equation. However, in order to get a better insight into the problem considered, in what follows, the square term in the equation is taken into account and thus, the vertical displacement is described by the following formula

$$\begin{aligned} v(Z^\alpha,t) =& f(Z^1,t) + u(Z^1,t)h'(Z^1) \\ &+ \frac{w(Z^1,t)}{H-h(Z^1)}\left[Z^2 - h(Z^1)\right], \end{aligned} \tag{4}$$

with

$$f = \frac{1}{2}h''(Z^1)\, u^2(Z^1,t). \tag{5}$$

For a constant bottom slope one should substitute $f = 0$ in Eq. (4).

Having the displacement components $u(Z^1,t)$ and $v(Z^1,Z^2,t)$ we can calculate the Jacobian of the transformation (1)

$$J = \det\left[z^i_\alpha\right] = (1+u')\left(1+\frac{w}{H-h}\right) \tag{6}$$

For incompressible fluids the Jacobian is equal to one, and thus

$$w(Z^1,t) = -(H-h)\frac{u'}{1+u'}. \tag{7}$$

Substitution of the relation into Eq. (4) leads to the following result

$$v(Z^\alpha,t) = u(Z^1,t)h' - \frac{u'}{1+u'}(Z^2-h) + f(Z^1,t). \tag{8}$$

With respect to the displacement component, the vertical velocity reads

$$\dot v(Z^\alpha,t) = \dot u(Z^1,t)h' - \frac{\dot u'}{(1+u')^2}(Z^2-h) + \dot f(Z^1,t). \tag{9}$$

Hereinafter the dots denote differentiation with respect to time.

Knowing the displacement field we can calculate the potential energy of the fluid

$$E_{pot.} = \rho g \int_0^L \int_h^H z^2(Z^\alpha,t) J dZ^2 dZ^1, \tag{10}$$

where ρ is the fluid density, g is the gravitational acceleration, $L = L_1 + L_2$ (see Fig. 1) and $J = 1$.

Substituting (1) and (8) into Eq. (9) and making integration over the fluid depth, one obtains

$$\begin{aligned} E_{pot.} = \frac{1}{2}\rho g H \int_0^L \Big[& H(1-\alpha^2) + 2h'(1-\alpha)u \\ & -H(1-\alpha)^2\frac{u'}{1+u'} + (1-\alpha)f \Big] dZ^1, \end{aligned} \tag{11}$$

in which α depends only on Z^1

$$\alpha = \alpha(Z^1) = \frac{h(Z^1)}{H}. \tag{12}$$

At the same time, the kinetic energy of the fluid is described by the formula

$$E_{kin.} = \frac{1}{2}\rho \int\limits_0^L \int\limits_h^H \left[(\dot{u})^2 + (\dot{v})^2\right] J dZ^2 dZ^1. \qquad (13)$$

From substitution of the velocity component (9) into the last equation and integration along the water depth, the following relation results

$$\begin{aligned}E_{kin.} = \frac{1}{2}\rho H \int\limits_0^L \Bigg[&(1+h'^2)(1-\alpha)(\dot{u})^2 - Hh'(1-\alpha)^2 \\ &\times \frac{\dot{u}\dot{u}'}{(1+u')^2} + \frac{1}{3}H^2(1-\alpha)^3\frac{(\dot{u}')^2}{(1+u')^4} + (1-\alpha)(\dot{f})^2 \\ &+ 2h'(1-\alpha)\dot{u}\dot{f} - H(1-\alpha)^2\frac{\dot{u}'\dot{f}}{(1+u')^2}\Bigg] dZ^1.\end{aligned} \qquad (14)$$

The relevant momentum equation is obtained by means of a standard variational procedure. For the conservative system considered the variation of the action integral reads

$$\delta I = \delta \int\limits_0^{t_k} (E_{kin.} - E_{pot.})dt. \qquad (15)$$

Knowing that the operations of variations and differentiations are commutative, the variation of the integrand variables in the equation gives

$$\begin{aligned}\delta I = &\frac{1}{2}\rho H \int\limits_0^{t_k}\int\limits_0^L \left[R_0\delta u + R_1\delta\dot{u} + R_2\delta u' + R_3\delta\dot{u}'\right] dZ^1 dt \\ &+ \frac{1}{2}\rho g H^2 \int\limits_0^{t_k}\int\limits_0^L \left[G_7\delta u' - \frac{1-\alpha}{H}(2h' + uh'')\delta u\right] \\ &\times dZ^1 dt = 0.\end{aligned} \qquad (16)$$

The terms of the integrands are

$$\begin{aligned}R_0 &= F_1(h'')^2 + F_2h'h'' - F_3Hh'', \\ R_1 &= 2(1+h'^2)G_1 - Hh'G_3 + F_4(h'')^2 + F_5h'h'' - F_6Hh'', \\ R_2 &= 2Hh'G_2 - \frac{4}{3}H^2G_6 + F_7Hh'', \\ R_3 &= \frac{2}{3}H^2G_5 - Hh'G_4 - F_8Hh'',\end{aligned} \qquad (17)$$

and

$$\begin{aligned}&F_1 = 2(1-\alpha)(\dot{u})^2u, \;\; F_2 = 2(1-\alpha)(\dot{u})^2, \\ &F_3 = (1-\alpha)^2\frac{\dot{u}\dot{u}'}{(1+u')^2}, \;\; F_4 = 2(1-\alpha)(u)^2\dot{u}, \\ &F_5 = 4(1-\alpha)u\dot{u}, \;\; F_6 = (1-\alpha)^2\frac{u\dot{u}'}{(1+u')^2}, \\ &F_7 = 2(1-\alpha)^2\frac{u\dot{u}\dot{u}'}{(1+u')^3}, \;\; F_8 = (1-\alpha)^2\frac{\dot{u}u}{(1+u')^2},\end{aligned} \qquad (18)$$

$$\begin{aligned}&G_1 = (1-\alpha)\dot{u}, \;\; G_2 = (1-\alpha)^2\frac{\dot{u}\dot{u}'}{(1+u')^3}, \\ &G_3 = (1-\alpha)^2\frac{\dot{u}'}{(1+u')^2}, \;\; G_4 = (1-\alpha)^2\frac{\dot{u}}{(1+u')^2}, \\ &G_5 = (1-\alpha)^3\frac{\dot{u}'}{(1+u')^4}, \;\; G_6 = (1-\alpha)^3\frac{(\dot{u}')^2}{(1+u')^5}, \\ &G_7 = (1-\alpha)^2\frac{1}{(1+u')^2}.\end{aligned}$$

For a constant bottom slope $h'' = 0$, $R_0 = 0$ and all the terms F_1, F_2, ... , F_7 are equal to zeros. For the linear operations considered, the terms in integrands (16) may be transformed to another forms. For example

$$R_1\delta\dot{u} = \frac{\partial}{\partial t}(R_1\delta u) - \dot{R}_1\delta u. \qquad (19)$$

Similar relations can be obtained for the remaining terms. In view of the relation, the variation of the action integral leads to the equation

$$\begin{aligned}&-\int\limits_0^{t_k}\int\limits_0^L \\ &\times\left[-R_0 + \dot{R}_1 + R_2' - \dot{R}_3' + gHG_7' + g(1-\alpha)(2h' + h''u)\right] \\ &\times\delta u\, dZ^1dt + \left. R_3\delta u\,\right|_0^{t_k}\Big|_0^L + \int\limits_0^L \left[R_1 - R_3'\right]\delta u|_0^{t_k} dZ^1 + \int\limits_0^{t_k} \\ &\times\left[R_2 - \dot{R}_3 + gHG_7\right]\Big|_0^L \delta u\, dt = 0.\end{aligned} \qquad (20)$$

For the discussed case of fluid motion starting from rest, the arbitrary variation δu vanishes at the end time points, i.e. for $t = 0$ and $t = t^k$. At the same time, we require (20) to vanish for all $\delta u(Z^1, t)$, which implies

$$-R_0 + \dot{R}_1 + R_2' - \dot{R}_3' + gHG_7' + g(1-\alpha)(2h' + h''u) = 0. \qquad (21)$$

For a prescribed generator motion $\delta u|_{Z^1=0} = 0$ and the last term in Eq. (20) gives

$$\left[R_2 - \dot{R}_3 + gHG_7\right]\Big|_{Z^1=L} = 0. \qquad (22)$$

One can check that the last condition if fulfilled at the shore point $Z^1 = L$ identically. Equation (21) is the momentum equation for the assumed motion of the fluid. From substitution of the descriptions (17) and (18) into (21) the following equation results

$$\begin{aligned}&-\frac{2}{3}H^2(1-\alpha)^2 \\ &\times\left[\ddot{u}''(1+u')^2 - 4\ddot{u}'u''(1+u') - 4\dot{u}'\dot{u}''(1+u') + 10(\dot{u}')^2u''\right] \\ &+ 2Hh'(1-\alpha)\left[\ddot{u}'(1+u')^2 - 2(\dot{u}')^2(1+u')\right] + 2Hh'(1-\alpha) \\ &\times\left[(\dot{u}')^2 - \ddot{u}u''\right](1+u')^3 + \\ &-\left[2(h')^2 - Hh''(1-\alpha) - 2(1+h'^2)(1+u')^2\right](1+u')^4\ddot{u} + \\ &-2gH\left\{(1-\alpha)u''(1+u')^3 + \frac{h'}{H}\left[1-(1+u')^2\right](1+u')^4\right\} + \\ &-2(\dot{u})^2\left[u(h'')^2 + h'h''\right](1+u')^6 + (1-\alpha)Hh''\dot{u}\dot{u}'(1+u')^4\end{aligned}$$

$$+\left[2(h'')^2\frac{\partial}{\partial t}(u^2\dot{u})+4h'h''\frac{\partial}{\partial t}(u\dot{u})\right](1+u')^6+$$
$$-Hh''(1-\alpha) \tag{23}$$
$$\times\left[\frac{\partial}{\partial t}(u\dot{u}')(1+u')^4-2u(\dot{u}')^2(1+u')^3\right]$$
$$+2Hh''(1-\alpha)\left[\frac{\partial}{\partial Z^1}(u\dot{u}\dot{u}')(1+u')^3-3u\dot{u}\dot{u}'u''(1+u')^2\right]$$
$$-4h'h''u\dot{u}\dot{u}'(1+u')^3+Hh''(1-\alpha)$$
$$\times\left[\frac{\partial}{\partial Z^1}(\ddot{u}u+\dot{u}^2)(1+u')^4-2(\ddot{u}u+\dot{u}^2)u''(1+u')^3+\right.$$
$$\left.-2\,\frac{\partial}{\partial Z^1}(u\dot{u}\dot{u}')(1+u')^3+6u\dot{u}\dot{u}'u''(1+u')^2\right]+$$
$$-2h'h''\left[\frac{\partial}{\partial t}(u^2u)(1+u')^4-2u\dot{u}\dot{u}'(1+u')^3\right]$$
$$+gh''u(1+u')^6=0.$$

In the case of a constant bottom slope $h''=0$, and the equation reduces to a simpler form. It is perhaps worth to add here that for the case shown in Fig. 1, the last condition is satisfied for all points $(0\leqslant Z^1\leqslant L)$ except for a small vicinity of the point B in the figure.

3. Approximate solutions to the momentum equation

The momentum Eq. (23) is the non-linear partial differential equation with respect to the independent variables $Z^1(0\leqslant Z^1\leqslant L)$ and $t\geqslant 0$. The equation has been derived under assumption that the slope of the fluid bottom is a small quantity ($|h'|<<1$). With respect to the assumption, we resort to approximate solution in which the displacement $u(Z^1,t)$ possess the power series expansion with respect to a small parameter ε ([7,8])

$$u=\varepsilon u_1+\varepsilon^2u_2+\varepsilon^3u_3+\cdots \tag{24}$$

In analysis developed in the further part we do not identify explicitly the parameter of the expansion. Substituting (24) into Eq. (23) and collecting terms with the same power of the parameter, a system of linear equations is obtained. In order to simplify the discussion we limit our consideration to the two lowest powers of the expansion. The first order approximation of the equation is

$$\left[1+\frac{1}{2}Hh''(1-\alpha)\right]\ddot{u}_1-\frac{1}{3}H^2(1-\alpha)^2\ddot{u}_1''$$
$$+Hh'(1-\alpha)\ddot{u}_1'-gH(1-\alpha)u_1''+g(2h'u_1'+\frac{1}{2}h''u_1)=0. \tag{25}$$

For $h=0$ the equation reduces to the case of constant depth. Similarly, the second power terms in the expansion lead to the following equation

$$\left[1+\frac{1}{2}Hh''(1-\alpha)\right]\ddot{u}_2-\frac{1}{3}H^2(1-\alpha)^2\ddot{u}_2''+Hh'(1-\alpha)\ddot{u}_2'$$
$$-gH(1-\alpha)u_2''+g(2h'u_2'+\frac{1}{2}h''u_2)-NL=0, \tag{26}$$

where the NL term in the equation depends on the first order solution:

$$NL=\frac{2}{3}H^2(1-\alpha)^2\left[\ddot{u}_1''u_1'-2\ddot{u}_1'u_1''-2\dot{u}_1'\dot{u}_1''\right]-Hh'(1-\alpha)$$
$$\times\left[2\ddot{u}_1'u_1'-(\dot{u}')^2-\ddot{u}_1u_1''\right]-2\left[3+Hh''(1-\alpha)+(h')^2\right]$$
$$\times\,\ddot{u}_1u_1'+3gH(1-\alpha)u_1'u_1''-9gh'(u_1')^2$$
$$-h'h''\left[(\dot{u}_1)^2+2u_1\ddot{u}_1\right]$$
$$-\frac{1}{2}Hh''(1-\alpha)\left[\ddot{u}_1u_1'+2\dot{u}_1\dot{u}_1'\right]-3gh''u_1u_1'. \tag{27}$$

Although the equations are linear, they have variable coefficients and thus, they are still difficult to be solved analytically. Therefore, in order to get solutions of the equations we resort to a discrete formulation by means of the finite difference method. With the discrete approach, the space derivatives with respect to the independent variable Z^1 are substituted by finite difference quotients, according to the formulae:

$$\frac{\partial u}{\partial Z^1}\cong\frac{1}{2a}\left(u_{j+1}-u_{j-1}\right),$$
$$\frac{\partial^2u}{\partial(Z^1)^2}\cong\frac{1}{a^2}\left(u_{j-1}-2u_j+u_{j+1)}\right), \tag{28}$$

where a is a constant spacing of horizontal nodal points $Z_j^1=j\cdot a$ ($j=0,1,2,\cdots,N$) with the end points $Z_0^1=0$ and $Z_N^1=L-a$, respectively.

In order to save the place, hereinafter we omit the lower indices of the dependent variables in Eqs. (25) and (26). For a typical point $k(Z^1=ka)$ within the fluid domain, the finite difference analogue of Eq. (25) is written as continuous in time and discrete in space equation

$$-W_1\ddot{u}_{k-1}+W_2\ddot{u}_k-W_3\ddot{u}_{k+1}-S_1u_{k-1}+S_2u_k-S_3u_{k+1}=0, \tag{29}$$

in which

$$W_1=\frac{1}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2+\frac{1}{2}h'\frac{H}{a}(1-\alpha),$$
$$W_2=1+\frac{1}{2}Hh''(1-\alpha)+\frac{2}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2, \tag{30}$$
$$W_3=\frac{1}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2-\frac{1}{2}h'\frac{H}{a}(1-\alpha),$$

and

$$S_1=\frac{1}{a}\left[\frac{gH}{a}(1-\alpha)+gh'\right],$$
$$S_2=2\frac{gH}{a^2}(1-\alpha)+\frac{1}{2}gh'', \tag{31}$$
$$S_3=\frac{1}{a}\left[\frac{gH}{a}(1-\alpha)-gh'\right].$$

Such equations are written for all the consecutive points: $Z^1=a,Z^1=2a,\cdots,Z^1=L-a$. A remark is needed. With respect to the discrete formulation we also need information about the displacement $u(t)$ of the corner point $Z^1=L$ (the shore point C in Fig. 1). In a formal way, a relevant differential equation for the boundary point may be obtained by taking the limit $Z^1\to L$ ($\alpha\to 1$) in Eq. (25). In the discrete model considered a better way however is to describe the displacement of the point by means of displacements of neighbouring nodal points. For instance, the displacement of the shore point may be assumed as equal to the displacement of the next point $Z^1 = L - a$. Another way is to express the displacement by means of the Gregory-Newton extrapolation formula [9]

$$u_C = \frac{11}{3}u_N - 5u_{N-1} + 3u_{N-2} - \frac{2}{3}u_{N-3}. \tag{32}$$

Numerical tests show that the two formulations lead to practically the same results except for a small vicinity of the corner point where differences between the two formulations do not exceed a few percentages. The final set of Eqs. (29) is written in the matrix form

$$[\mathbf{AM}]\ (\ddot{\mathbf{U}}) + [\mathbf{BM}]\ (\mathbf{U}) + (\mathbf{P}) = \mathbf{0}. \tag{33}$$

In the equation

$$\begin{aligned}
&(\mathbf{U})^T = (u_1,\ u_2,\ \cdots,\ u_N),\\
&(\ddot{\mathbf{U}})^T = (\ddot{u}_1,\ \ddot{u}_2,\ \cdots,\ \ddot{u}_N),\\
&(\mathbf{P})^T = (-W_1\ddot{u}_0 - S_1 u_0,\ 0,\ 0,\ \cdots,\ 0).
\end{aligned} \tag{34}$$

In accordance with the extrapolation (32) the matrix $[\mathbf{AM}]$ assumes the following form

$$[\mathbf{AM}] = \begin{bmatrix} W_2 & -W_3 & & & & \\ -W_1 & W_2 & -W_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & -W_1 & W_2 & -W_3 \\ & & W_4^\bullet & W_3^\bullet & -W_1^\bullet & W_2^\bullet \end{bmatrix}, \tag{35}$$

where:

$$\begin{aligned}
&W_1^\bullet = W_1^N - 5W_3^N, \quad W_2^\bullet = W_2^N - \frac{11}{3}W_3^N,\\
&W_3^\bullet = -3W_3^N, \quad W_4^\bullet = \frac{2}{3}W_3^N.
\end{aligned} \tag{36}$$

The superscript N in the equations denotes the nodal point $Z^1 = L - a$. In a similar way, the matrix $[\mathbf{BM}]$ reads

$$[\mathbf{BM}] = \begin{bmatrix} S_2 & -S_3 & & & & \\ -S_1 & S_2 & -S_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & -S_1 & S_2 & -S_3 \\ & & S_4^\bullet & S_3^\bullet & -S_1^\star & S_2^\star \end{bmatrix}, \tag{37}$$

with

$$\begin{aligned}
&S_1^\bullet = S_1^N - 5S_3^N, \quad S_2^\bullet = S_2^N - \frac{11}{3}S_3^N,\\
&S_3^\bullet = -3S_3^N, \quad S_4^\bullet = \frac{2}{3}S_3^N.
\end{aligned} \tag{38}$$

It should be noted that the non-zero elements of the matrices depend on the independent variable Z^1 of the considered point but do not depend on time. It may be seen that the first equation of the set (29) contains terms $\ddot{u}_0$ and u_0 which are known functions of time as boundary condition at $Z^1 = 0$. In order to perform integration of Eq. (29) in the time domain, we introduce the discrete time and make use of the Wilson θ method. In this method, the acceleration between the subsequent time steps is approximated by a linear function of time. For a mechanical system, the procedure is unconditionally stable for $\theta > 1.37$ [10]. In order to make the discussion clear some fundamental equations of the method are summarised below (for details see [10]). Assuming that we know solution of the problem at the time t, the standard equations of the method are

$$\begin{aligned}
&\dot{u}_{(3)} = \frac{3}{DT}\left(u_{(3)} - u_{(1)}\right) - 2\dot{u}_{(1)} - \frac{DT}{2}\ddot{u}_{(1)},\\
&\ddot{u}_{(3)} = \frac{6}{DT^2}\left(u_{(3)} - u_{(1)}\right) - \frac{6}{DT}\dot{u}_{(1)} - 2\ddot{u}_{(1)},
\end{aligned} \tag{39}$$

where $u_{(1)} = u(t)$, $u_{(3)} = u(t + DT)$, and DT = $\theta\Delta t$ with $\theta = 1.47$.

The second of Eqs. (39) may be rewritten as

$$\frac{DT^2}{6}\ddot{u}_{(3)} = u_{(3)} - \left[u_{(1)} + DT\ \dot{u}_{(1)} + \frac{DT^2}{3}\ddot{u}_{(1)}\right]. \tag{40}$$

The relation holds for each nodal point, and thus, the system of Eqs. (33) written for the time $t_{(3)} = t_{(1)} + DT$ assumes the form

$$\begin{aligned}
&\left([\mathbf{AM}] + \frac{DT^2}{6}[\mathbf{BM}]\right)\ (\mathbf{U}_{(3)})\\
&- [\mathbf{AM}]\ (\mathbf{UA}) + \frac{DT^2}{6}(\mathbf{P}) = \mathbf{0}\,,
\end{aligned} \tag{41}$$

where

$$(\mathbf{UA}) = (\mathbf{U}_{(1)}) + DT\,(\dot{\mathbf{U}}_{(1)}) + \frac{DT^2}{3}(\ddot{\mathbf{U}}_{(1)}) \tag{42}$$

is a known solution for the time level $t_k = t_{(1)}$, and $(\mathbf{P})$ describes the generator motion.

For the initial value problem considered it is reasonable to assume a smooth beginning of the fluid motion starting to move from rest, for which not only the velocity, but also the acceleration field disappear at the initial moment of time. In order to describe such a generation of the fluid motion w apply here results developed in [11]. The horizontal motion of the piston – type generator is described by the formula

$$u_0(t) = a \cdot A(\tau)\cos\omega\, t + D(\tau)\sin\omega\, t, \tag{43}$$

where ω is the angular frequency, $a = s^{-3}$ is a time factor, and

$$\begin{aligned}
&A(\tau) = \frac{1}{3!}\tau^3\exp(-\tau),\\
&D(\tau) = 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right)\exp(-\tau), \tau = \eta t.
\end{aligned} \tag{44}$$

The parameter η in the relations is responsible for a growth in time of the generator amplitude. With passing time the generation approaches the harmonic generation with unit amplitude and the prescribed angular frequency. Having the formulae one can calculate the displacement, velocity and the acceleration on the boundary $Z^1 = 0$ needed in our procedure at each step of the discrete time. The solution of the Eq. (41) is obtained with the help of a standard procedure for linear algebraic system of equations. Knowing the solution at $t + DT$, it is a simple task to calculate the state of the system at the subsequent moment of time, i.e. at $t + \Delta t$

$$\begin{aligned} &\ddot{u}(t+\Delta t) = \ddot{u}_{(1)} + \frac{\Delta t}{DT}(\ddot{u}_{(3)} - \ddot{u}_{(1)}), \\ &\dot{u}(t+\Delta t) = \dot{u}_{(1)} + \Delta t \ddot{u}_{(1)} + \frac{(\Delta t)^2}{2DT}(\ddot{u}_{(3)} - \ddot{u}_{(1)}), \\ &u(t+\Delta t) = u_{(1)} + \Delta t \dot{u}_{(1)} + \frac{(\Delta t)^2}{2}\ddot{u}_{(1)} \\ &+ \frac{(\Delta t)^3}{6DT}(\ddot{u}_{(3)} - \ddot{u}_{(1)}). \end{aligned} \tag{45}$$

4. First order solution

In order to learn more about main features of the discrete model discussed above, let us consider the first order approximation to the momentum equation. It may be seen from Eq. (29), that the discrete model does not contain damping terms. At the same time, for the harmonic generation of the fluid motion a resonance phenomena may occur. In order to examine the case, let us consider the homogeneous system

$$[\mathbf{AM}]\ (\ddot{\mathbf{U}}) + [\mathbf{BM}]\ (\mathbf{U}) = \mathbf{0} \tag{46}$$

obtained by disregarding the forcing term $(\mathbf{P})$ in Eq. (33).

Substituting the time harmonic factor $\exp(i\omega t)$, i.e. $\mathbf{U} = \mathbf{U}_0 \exp(i\omega t)$ into the equation, one obtains

$$\left([\mathbf{BM}] - \omega^2 [\mathbf{AM}]\right)\ (\mathbf{U}_0) = \mathbf{0}. \tag{47}$$

The formula describes the eigenvalue problem for the matrices $[\mathbf{AM}]$ and $[\mathbf{BM}]$. A standard procedure enables us to calculate the eigenvalues for a chosen water depth on the generator face and a set of bottom slopes of the triangular fluid domain. For the problem considered the most important are the lowest eigenvalues. Numerical calculations have been performed for the depth $h = 0.60$ m on the left boundary and the bottom slopes equal to 0.05, 0.10, 0.15 and 0.20. In all the cases a discrete model with $N = 101$ nodal points has been considered. Some numerical results obtained in the computations are shown in Table 1.

Table 1
Eigenfrequencies of the fluid domain

Sequence number	$h' = 0.05$ $L_2 = 12$	$h' = 0.10$ $L_2 = 6$	$h' = 0.15$ $L_2 = 4$	$h' = 0.20$ $L_2 = 3$
1	0.3831	0.7641	1.1409	1.5117
2	0.7006	1.3923	2.0666	2.7154
3	1.0141	2.0039	2.9461	3.8200
⋮	⋮	⋮	⋮	⋮
100	11.6520	16.1542	19.4162	22.0132
101	11.8168	16.6032	20.2116	23.2042

The numbers in the table display the subsequent frequencies for each of the bottom slopes. As compared to the case of water waves propagating in fluid of uniform depth equal to 0.60 m, the lowest frequencies for the slope $h' = 0.1$ correspond to the waves of length: $\lambda_1 = 19.95$ m, $\lambda_2 = 10.95$ m and $\lambda_3 = 7.61$ m, respectively. Assuming that the waves amplitude equals $\eta_0 = \kappa H$, where $\kappa \in (0.05 \div 0.10)$ in our case, the relevant Ursell parameter for the waves $U_r = 40 \div 85$, and thus, such cases correspond to non-linear shallow water waves described by the Boussinesq and Korteweg–de Vries equations [12]. In the discussed problem however, we have a finite domain of fluid with variable depth and therefore the Ursell number is not a proper characteristic of the phenomenon. The remainder, higher frequencies in the table, correspond to shorter waves. The eigenvalues cover a relatively large range of frequencies associated with waves of different lengths and therefore, one should be almost certain that in a vicinity of a coastline, a fluid flow will fall in a resonance mode with waves arriving at the coastline. The result of such a phenomenon is a growth of water wave up till a breaking of it will be reached. In real conditions, due to dissipation of energy of waves approaching a beach, a motion of water near the shoreline is of course more complicated than the results of the model considered. Nevertheless, the important result of the model on the resonance phenomenon reflects a real feature of flow induced by periodic waves arriving at a coastline. It is perhaps of importance to emphasise here that, because of the changing water depth and the resonance phenomenon, the flow within the finite domain will, in general, not be periodic even for the periodic waves approaching the shore. In order to illustrate the phenomenon, numerical calculations have been performed for a set of angular frequencies of the generation. In particular, we have chosen a resonance frequency, and, for comparison, a frequency which does not belong to the resonance set. In all cases considered the amplitude of the generator motion was assumed to be constant and equal to 0.06 m.

In the numerical procedure, as a first step, horizontal displacements of chosen material points in the vicinity of the shoreline are calculated as functions of a discrete time sequence. Then, in the second step, the space derivatives of the horizontal displacements are evaluated. The vertical displacements are obtained with the help of Eq. (7). Some of the results obtained in numerical calculations are shown in Fig. 2, where the graphs illustrate the first order solution of the problem mentioned.

The plots describe the displacements of the point $L-a$. The relevant displacements of the shoreline point are nearly equal to that shown in the figure. From the plots it may be seen, that, within the resonance range, the model behaves like a classical linear mass - springs system loaded with a harmonic force of frequency equal to an eigenfrequency of the system. The linear model enables us to calculate a run up of the shoreline. It is seen that the displacements of the line exceed the amplitude of the generation significantly. The last feature is mainly a result of the diminishing water depth, however in the case of a resonance, one may expect additional growth of the displacements. In order to evaluate the vertical displacement of the fluid particles forming the free surface we need to calculate the space derivative of the horizontal displacement.

The amplitude of the derivative grows when going to the shoreline. And thus, it may happen, that the assumption that $|u'|$ is a small quantity is satisfied only in a deeper part of the fluid, in a certain distance from the coastline, and fails to be valid within the range of the smallest water depth. More precisely, because of the denominator $(1+u')$ of Eq. (7), proper results of vertical displacements can be obtained only for cases when the denominator is far off zero. In the case the derivative goes to minus one the formula leads to an indeterminate result. For the latter case, the vertical displacement may be obtained from the approximated formula $v \cong h'u$, which delivers plausible results for the range of smallest water depth – in the vicinity of the coastline C in Fig. 1. But, even in the singular case $u' = -1$, the model presented above gives proper horizontal displacements, which carry important information about the phenomenon. Therefore, although the formula (7) may be not valid within the range of the smallest water depth we leave it as the theoretical result of the model within the whole range $0 \leqslant Z^1 \leqslant L$. It means that as far as the horizontal displacements of the fluid are concerned, the model provides reliable results, but fails to deliver similar results for the vertical component of the displacement.

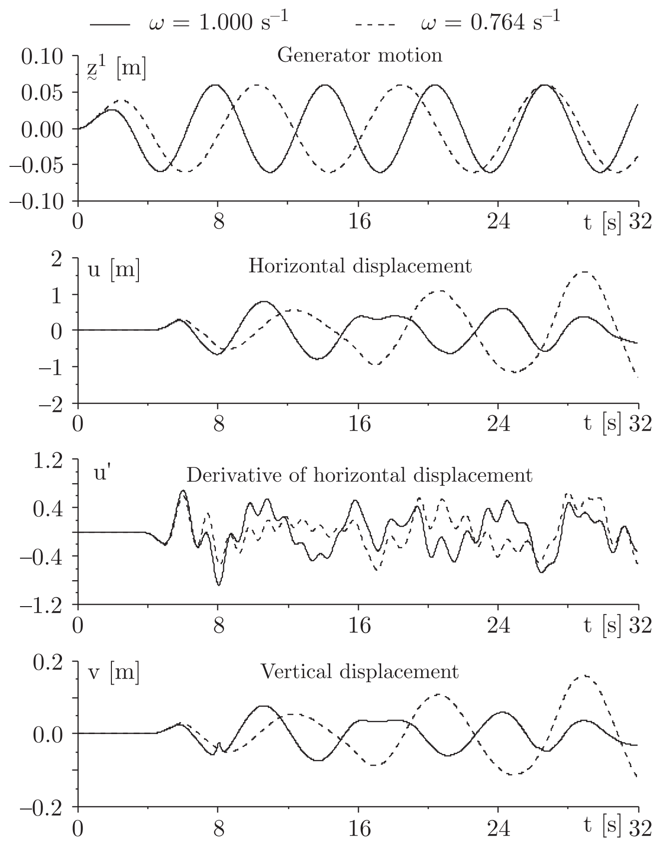

Fig. 2. Linear solution for the point $L-a$

The results obtained enable us to calculate the free surface elevation at chosen space point i.e. $\eta(z^1 = \text{const.}, t)$. Denoting by z_c the space point we transform Eqs. (1) and (7) into the following form

$$Z^1 + u(Z^1,t) = z_c,$$

$$\eta(z_c,t) = h'(Z^1)u(Z^1,t) - \frac{u'(Z^1,t)}{1+u'(Z^1,t)}\left[H - h(Z^1)\right]. \tag{48}$$

Having a solution of Eq. (25) we can find a solution of the last equations by means of an iterative procedure. With respect to the latter description, one must remember that at the space point the depth of the fluid should be greater than zero. Otherwise, the shoreline point C (see Fig. 1.) will occupy a position below the considered space point within a certain range of time.

5. Second order solution

Having the first order solution of the momentum equation, one can calculate the second order term, and accordingly, the second order solution. The latter can be obtained by integration of Eq. (26) with respect to time and space.

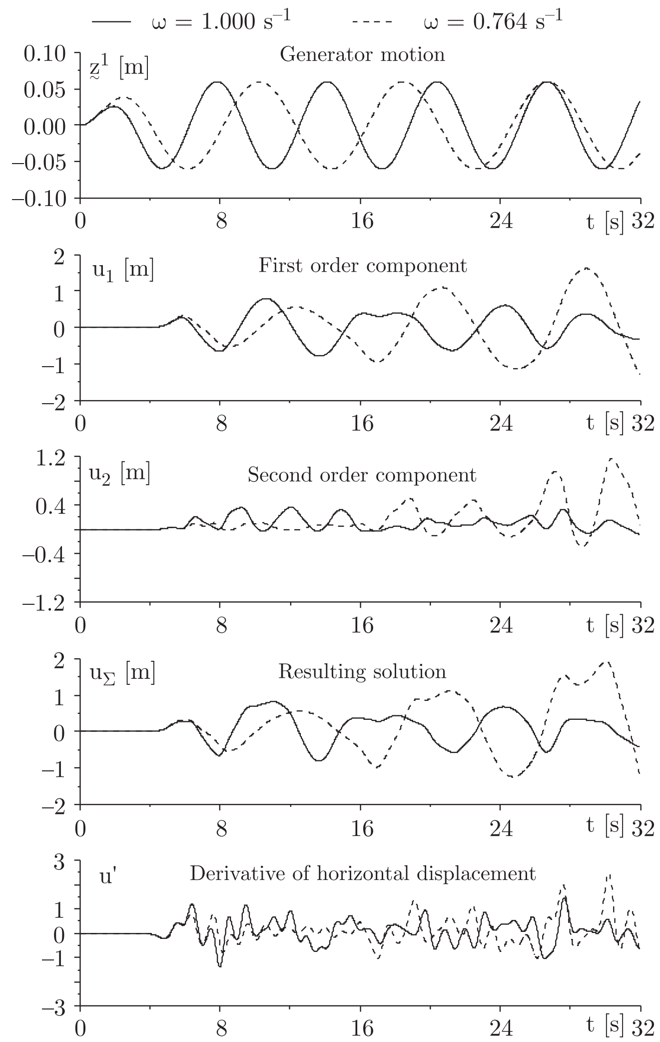

Fig. 3. Non-linear solution for the point $L-a$

Like in the previous case we have the linear partial differential equation with variable coefficients, but now, the free term of the equation depends on the first order result. Like in the previous case numerical calculations have been made for the bottom slope $h' = 0.1$ and the frequencies $\omega = 1$ s^{-1} and $\omega = 0.7641$ s^{-1} of the generation. Some of the results obtained in numerical calculations are shown Fig. 3, where the graphs illustrate main features of the model at hand. From the graphs it follows that average distributions in time of the second order terms are greater than zero. It means, that much information about bottom slope in the formulation is carried by the second order term. In the case of a resonance both of the components are reinforced in time.

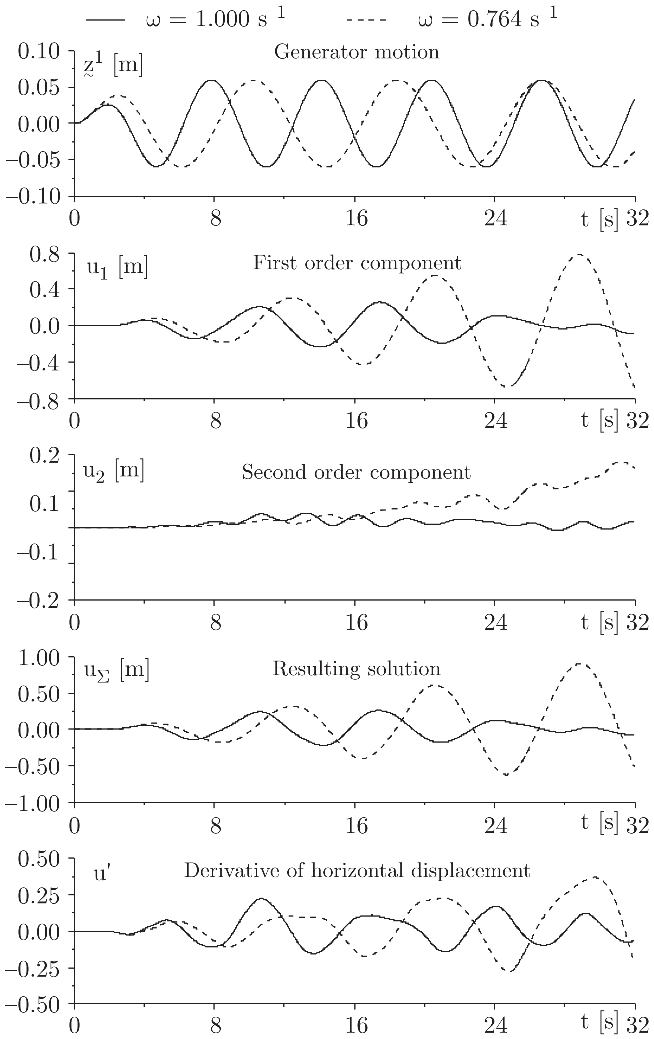

Fig. 4. Non – linear solution for the point $L - L_2/3$

At the same time, it may be seen that, as compared to the generator motion, the resultant space derivative $u'(Z^1 = Z^1_N, t)$ has a relatively complex distribution in time. Moreover, the derivative is not a small number and in some ranges of time is close to minus one. On the other hand, one can expect less complex distributions of the variables for deeper part of the fluid. An illustration of the latter case is presented in Fig. 4, where the plots represent the solution for the material points $Z^1 = L - L_2/3$. From the plots in Fig. 3 it follows that vertical component of the displacement field in the smallest water depth cannot be obtained from the formula (7). The component may be estimated directly from the horizontal displacement. Such approximation is justified for shallow water and may be good enough for practical purposes. On the other hand, from practical point of view, the most important are horizontal displacements, which are described with a reasonable accuracy by the model considered. Although we have confined our attention to the non-breaking waves, the formulation for finite fluid domains provides useful estimation of the run up of shoreline also for breaking waves.

6. Concluding remarks

The motion of water in neighbourhood of a sloping beach depends mainly on bottom slope and characteristics of water waves approaching the beach. Since the water depth diminishes towards the beach, one can observe a significant change of the wave height in this zone. For many cases, due to the growing steepness of the waves, the waves lose their stability and a breaking phenomenon occurs. The growth of the wave height is induced by the change of water depth and a resonance phenomenon of the fluid motion. The latter takes place when a frequency of approaching waves is close to frequency of waves reflected from the beach. In such a case a collision of the incoming waves with the reflected waves may reinforce the waves height. In a general case, in order to describe a motion of the fluid in such a domain it is necessary to find a solution of non-linear partial differential equations with non-linear boundary conditions on moving boundaries. Since we have no closed analytical solution of the equations, we are forced to resort to certain approximations of the equations. Commonly, the equations are substituted by the so-called non-linear equations for shallow water with space and time co-ordinates as independent variables. But, even in the latter case it is difficult to find a solution of the equations. In the present work, in a theoretical description of the problem, we apply material and time co-ordinates as independent variables. With the latter approach it is much easier to solve boundary conditions on a shoreline. In particular, with assumed displacements of fluid particles, the problem has been reduced the one-dimensional in space, time dependent model describing the fluid motion. In order to simulate water waves approaching a beach a 'triangular' fluid domain has been considered with motion induced by a piston type wave-maker. Basic equations of the model have been derived by means of a variational procedure. The non-linear equations, obtained in this way, have been substituted by a system of equations resulting from power series expansions with respect to a small parameter. With accordance to the procedure mentioned, the most important is the linear, first order approximation of the equations. The linear momentum equation carries information on the water depth, as well as on the resonance phenomenon. Formulation of the problem in the discrete space has led to the eigenvalue problem for two matrices representing the momentum equations for a set of nodal material points. From the analysis it follows that the eigenvalues cover a relatively large range of frequencies inherent for water waves, and therefore, a resonance phenomenon of the waves appears. It should be stressed however, that, in a general non-linear case, it is not possible to separate the non-linear influence from the resonance phenomenon. Numerical integration of the linear momentum equation in the time domain has confirmed that, the generation of water flow with the resonance frequency will result in growing water wave height. The system behaves like a mass - spring system loaded with harmonic force within a resonance range. In order to obtain vertical component of the displacement field, we need to calculate the space derivative $\partial u/\partial Z^1$. Because of the oscillating behaviour of the horizontal displacement $u(Z^1, t)$ in the neighbourhood of a shoreline, it may happen that the derivative is close to minus one and, at a certain moment of time the relevant vertical displacement becomes undefined. Such a case indicates a breaking phenomenon. An approximate evaluation of the vertical displacement for the case $u' \cong -1$ may be obtained directly from the horizontal displacement, by means of the approximate formula $v \cong h'u$. The latter approach is justified in the range of smallest water depth, say for $L - H \leqslant Z^1 \leqslant L$. It is important to note, that, even in the case of the breaking wave, the discrete model developed above enable us to calculate the horizontal displacement of a shoreline (the point C in Fig. 1).

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