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Linearization of non-linear state equation

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Abstract. The paper presents an overview of linearization methods of the non-linear state equation. The linearization is developed from the point of view of the application in the theoretical electrotechnics. Some aspects of these considerations can be used in the control theory. In particular the main emphasis is laid on three methods of linearization, i.e.: Taylor's series expansion, optimal linearization method and global linearization method. The theoretical investigations are illustrated using the non-linear circuit composed of a solar generator and a DC motor. Finally, the global linearization method is presented using several examples, i.e. the asynchronous slip-ring motor and non-linear diode. Furthermore the principal theorem concerning the BIBS stability (bounded-input bounded state) is introduced.

Key words: non-linear state equation, linearization, optimal linearization method, global linearization method, Taylor's series expansion, BIBS stability.

1. Introduction

The subject of the linearization of nonlinear state equation has been discussed in a number of papers concerning theoretical electrotechnics and control theory [1–8]. This problem involves ranges of equivalence of the linear model both in direct mapping of the systems dynamics as well as their stability, controllability and observability [2,5,9]. The most common linearization method i.e. expansion in Taylor's series around the equilibrium point is a very effective approximation of the non-linear model only for some minor deviation of state variables from the equilibrium point [4]. However, this method can be a good starting point for other methods that are good approximations in the whole state space [4,10,11]. In recent years a significant importance has had the linearization by variable transformation which is based on global diffeomorphism [6–8]. Its fundamental principles will be presented in a further part of this paper. It should be noted, however, that the continuity of the non-linear functions and their differentiability plays, in this case, the most significant role [6,7,12]. An interesting method of linearization was presented in paper [9], where non-linear state equation was approximated by linear state model with matrix $A = A(t)$. In this case the sequence of linear observers is uniformly convergent which results in an observer for a non-linear system. In paper [13] the scalar non-linear Bernoulli equation was also approximated by the linear model and it was found that there was a good agreement of the approximation series $\dot{x} = A[x(t)^n]x$, $(n = 1, 2, ...)$ with the numerical solution of the non-linear equation $\dot{x} = f(x, t)$. The linearization of the multi-input, multi-output systems (MIMO) by the input-output injection was presented in papers [3,14,15]. Works [6,7] present little known Frobenius theorem concerning the linearization of partial differential equations. Moreover, it should be stressed that Frobenius integrability of certain distributions associated to a control system is equivalent to its feedback linearizability. It should be also noted that a historical review of non-linear control methods, which also describes the linearization of non-linear systems, is presented in paper [16].

In this work in overview of the basic methods of the linearization of non-linear state equation is presented. The overview concerns basic problems of theoretical electrotechnics dealing with the linearization of the non-linear state equations. The problems presented here can be also used in control theory.

In many considerations concerning system dynamics the physical systems are treated as linear systems. This follows from assumed simplified statements that say that the characteristics of system elements are linear in character, or that the equation linearized by Taylor's expansion occurs for some small deviations of state variables around the equilibrium point. However, in many cases, it is impossible to accept such assumptions and for our analysis we assume the following system of non-linear equations.

$$\dot{x} = f(x, u, t), \quad x(0) = x_0 \tag{1}$$

where $f(x, u, t)$ is the vector of nonlinear functions, $x(t) \in R^n$ and $u(t) \in R^m$ are the vector of the state variables and the input vector, respectively; x_0 represents the set of initial conditions.

In practical considerations, to solve Eq. (1) we apply the numerical methods [17–20]. The problems of the solution of Eq. (1) are not examined exactly. Alike, the stability of the system that is described by Eq. (1) is the open problem [21,22].

For this reason, we approximate the Eq. (1) by linear state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \tag{2}$$

The form of the matrices A and B depends of the method of the linearization of Eq. (1). Equation (2) has the analytical

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solution

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)dt. \tag{3}$$

Assuming $t_0 = 0$, w obtain

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)dt. \tag{4}$$

The exact solution of Eq. (2) resulting from Eq. (3) or Eq. (4) is very important to solve the stability problems in the linear systems. However, in practice, for the number of state variables $n > 3$ we use the numerical methods to compute the vector $x(t)$.

The homogeneous state equation is given as follows

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0. \tag{5}$$

The solution of (5) takes the form

$$x(t) = e^{A(t-t_0)}x_0 \tag{6}$$

or

$$x(t) = e^{At}x_0, \quad \text{if} \quad t_0 = 0. \tag{7}$$

THEOREM 1. The system described by homogeneous Eq. (5) is asymptotically stable if and only if the eigenvalues of matrix A have the negative real parts [22].

In the case of the non-homogenous equation, we can formulate the following theorem:

THEOREM 2. The system described by non-homogenous Eq. (2) with input $u(t)$ is BIBS (bounded-input bounded state) stable if and only if the eigenvalues of matrix A have the negative real parts and the input $u(t)$ is limited [8].

This theorem is equitable in the case, where we do not meet the secular terms [23].

The linearization of non-linear state equation (1) aims to make the linear approach (2) a good approximation of the non-linear equation in the whole state space and for time $t \to \infty$. In the above case the linear approach can ensure the existence and an unambiguous solution for the non-linear equation. It can also constitute a mathematical model that makes it possible to investigate the stability of the non-linear system.

In this paper three linearization methods of the non-linear state equation are defined:
– expansion in Taylor's series,
– optimal linearization method,
– global linearization method.

The problem of the linearization based on the geometrical approach will be discussed in another paper.

To illustrate the above linearization methods we use the same non-linear electric circuit containing a DC motor supplied by a solar generator.

2. Non-linear electric circuit with a DC motor supplied by a solar generator

To illustrate the theoretical results developed for the three methods mentioned above, the non-linear electrical circuit with the solar generator and DC drive system is analysed. The non-linear circuit is presented in Fig. 1. In the time $t = 0$ the switch W is closed and the circuit is in a transient state. The non-linear characteristic of the solar generator is showed in Fig. 2 [24,25].

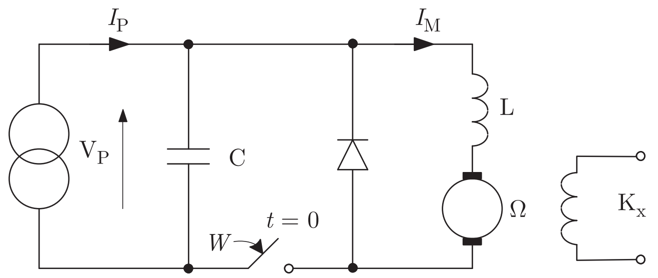

Fig. 1. An electric circuit containing a solar generator and a DC motor

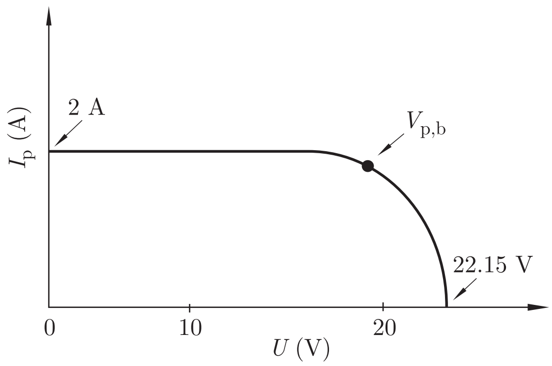

Fig. 2. Non-linear characteristic of solar generator. $V_{p,0} = 22,15$ V is the generator voltage prior to switching on switch W. $V_{p,b}$ is the equilibrium point of the system (steady state)

The transient state of the circuit is described by the following set of equations [24,25].

$$\begin{aligned} \dot{x}_1 &= -a_1 e^{ax_1} - a_2 x_2 + u \\ \dot{x}_2 &= a_3 x_1 - a_4 x_2 - a_5 x_3 \\ \dot{x}_3 &= a_6 x_2 - a_7 x_3; \\ x_1(0) = V_{p,0}, &\quad x_2(0) = 0, \quad x_3(0) = 0, \end{aligned} \tag{8}$$

where $x_1 = V_p$ is the generator voltage, $x_2 = I_M$ is the rotor current and $x_3 = \Omega$ represents the DC motor rotational speed. The non-linear characteristic of the solar generator is approximated using the following formula

$$I_p = I_0 - I_s(e^{aV_p} - 1). \tag{9}$$

In this formula I_0 is the photovoltaic current of the cell $(V_p = 0)$ dependent on light flux, I_s is the saturation current defined by Shockley equation, whereas a is the factor that characterizes the solar generator.

The coefficients $a_1, ..., a_7$ and u are expressed by the relations that combine the parameters of non-linear circuit (Figs. 1 and 2)

$$\begin{aligned} a_1 &= \frac{I_s}{C}, \quad a_2 = \frac{1}{C}, \quad a_3 = \frac{1}{L}, \quad a_4 = \frac{R_m}{L} \\ a_5 &= \frac{K_x}{L}, \quad a_6 = \frac{K_x}{J}, \quad a_7 = \frac{K_r}{J}, \quad u = \frac{I_0 + I_s}{C}. \end{aligned} \tag{10}$$

In the numerical computations, we use the following values of the parameters:

$$\begin{aligned} R_m &= 12.045\ \Omega, \quad L = 0.1\ \mathrm{H}, \quad C = 500\mu\mathrm{F}, \\ K_x &= 0.5\ \mathrm{Vs}, \quad K_r = 0.1\ \mathrm{Vs}^2, \quad J = 10^{-3}\ \mathrm{Ws}^3, \\ I_0 &= 2\ \mathrm{A}, \quad I_s = 1.28 \cdot 10^{-3}\ \mathrm{A}, \quad a = 0.54\ \mathrm{V}^{-1}, \\ V_{p,0} &= 22.15\ \mathrm{V}. \end{aligned} \tag{11}$$

The system of non-linear Eqs. (8) is solved using Runge-Kutta method [20] with the integration step $h = 10^{-6}$ s. The solution is presented in Figs. 3, 4 and 5.

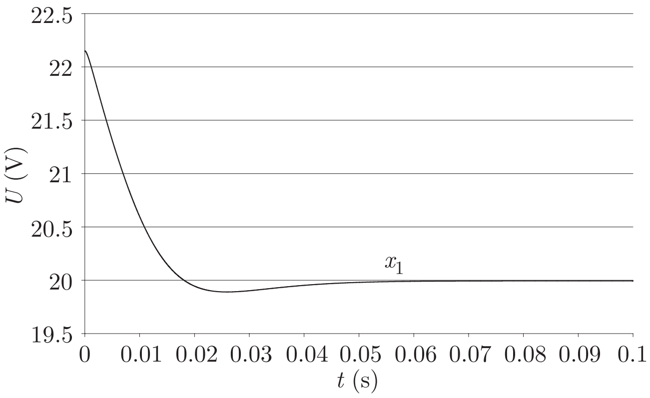

Fig. 3. The diagram of state variable x_1

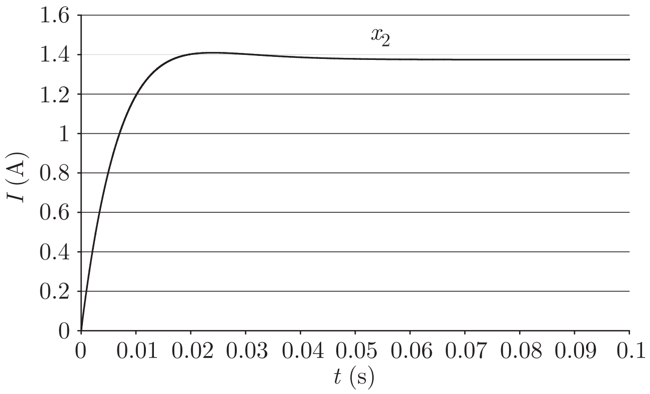

Fig. 4. The diagram of state variable x_2

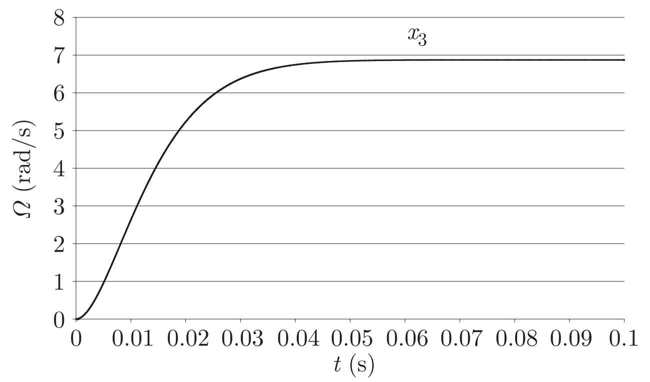

Fig. 5. The diagram of state variable x_3

3. Expansion in the Taylor's series

Let x_{eq}, u_{eq} be the equilibrium point of the system (1), i.e.

$$\dot{x}_{eq} = f(x_{eq}, u_{eq}, t) \tag{12}$$

and

$$\Delta x = x - x_{eq}, \quad \Delta u = u - u_{eq} \tag{13}$$

are the small differences for the state vector and the input vector, respectively. Assuming that

$$\Delta \dot{x} = \dot{x} - \dot{x}_{eq} = \dot{x} - f(x_{eq}, u_{eq}, t) \tag{14}$$

and expanding in Taylor's series the right side of Eq. (1), and neglecting the terms of order higher than first, we obtain the approximation of this equation in the form of the following linear equation

$$\Delta \dot{x} = A \Delta x + B \Delta u. \tag{15}$$

We usually write Eq. (15) in the form [4,26]

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{16}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_{eq}\\u=u_{eq}}}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_{eq}\\u=u_{eq}}}. \tag{17}$$

Now, we consider the electric circuit presented in Fig. 1. To simplify the linearization of the equation system (8) in Eq. (16) we use transient and steady components in our analysis

$$x(t) = x_s(t) + x_t(t). \tag{18}$$

The steady components $x_s(t)$ are computed from the set of non-linear algebraic equations

$$f(x_s(t), u_s(t)) = 0 \tag{19}$$

and the transient components $x_t(t)$ are the solution of homogeneous equations

$$\dot{x}_t - \dot{x}_{eq} = A(x_t - x_{eq}). \tag{20}$$

For the stable system we have $\dot{x}_{eq} = x_{eq} = 0$ and Eq. (20) is reduced to

$$\dot{x}_t = Ax_t(t), \quad x_t(0) = x(0) - x_s(0) \tag{21}$$

where matrix A is computed using both Eq. (17) and constant $x = x_{eq} = x_s$, $u = u_{eq}$. To illustrate this method we use the example described in Section 2. In this case the non-linear circuit is composed with the solar generator and DC motor.

The solutions of the non-linear equation using Runge-Kutta method with the integration step $h = 10^{-6}$ s and the solution of linear equation (16) are represented in Figs. 6–8. In this case we use the method of decomposition of the state variables on the steady components $x_s(t)$ and transient components $x_t(t)$, $(x(t) = x_s(t) + x_t(t))$. The equilibrium point is chosen in the steady state, i.e. in the point where $f(x_s(t), u_s(t), t) = 0$. This behaviour is named Taylor's series expansion around equilibrium point with the transient components.

It is possible to realize the expansion in Taylor's series around the initial condition $x(0) = x_0$. This procedure gives us immediately $x_{eq} = x_0$ and $\dot{x}_{eq} = \dot{x}(t = 0)$. This behaviour is very convenient for the case $x_0 = 0$.

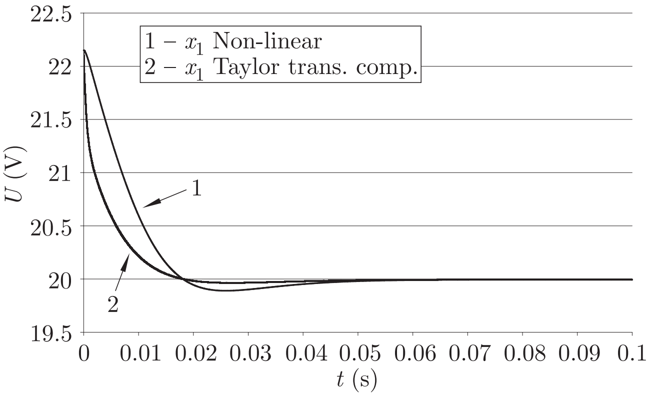

Fig. 6. The diagram of state variable x_1 – comparison between solution of non-linear Eq. (8) and solution of linear Eq. (16)

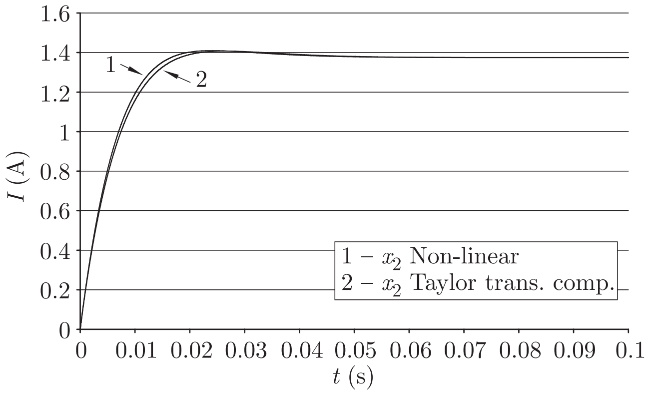

Fig. 7. The diagram of state variable x_2 – comparison between solution of non-linear equation (8) and solution of linear Eq. (16)

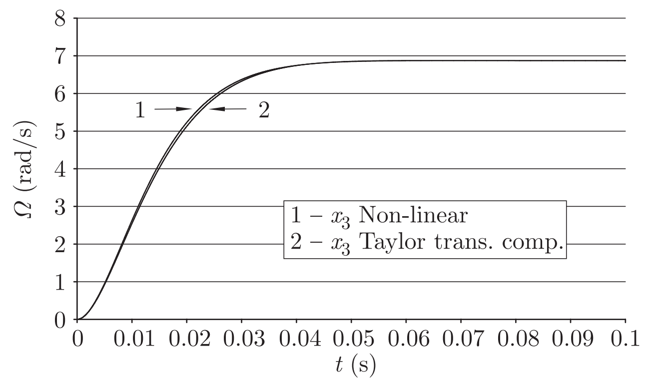

Fig. 8. The diagram of state variable x_3 – comparison between solution of non-linear equation (8) and solution of linear Eq. (16)

4. Optimal linearization method

The least square method makes it possible to find the method of linearization of Eq. (1) named the optimal linearization method [4,25,27]. In this case the non-linear equation is approximated by the optimal equation

$$\dot{x}(t) = A^* x(t) + B^* u(t), \quad x(0) = x_0. \tag{22}$$

The optimal matrices A^* and B^* are defined in the following way: the small difference

$$\varepsilon = Ax(t) + Bu(t) - f(x(t), u(t), t) \tag{23}$$

between the right side of linear and non-linear equation is thus defined.

Unknown elements a^*_{ij} and b^*_{ij} $(i, j = 1, 2, ..., n)$ of the matrices A^* and B^* are determined by minimizing of the functional

$$I(a_{ij}, b_{ij}) = \int_0^{t_1} \varepsilon^T(t)\varepsilon(t)dt \tag{24}$$

$$\left.\frac{\partial I(a_{ij}, b_{ij})}{\partial a_{ij}}\right|_{a_{ij} = a^*_{ij}} = 0 \, , \; \left.\frac{\partial I(a_{ij}, b_{ij})}{\partial b_{ij}}\right|_{b_{ij} = b^*_{ij}} = 0 \, . \tag{25}$$

To determine the optimal elements a^*_{ij} and b^*_{ij}, we introduce the basis functions into the formula (23). The basis functions can be defined using Taylor's series expansion of the non-linear equation (1). The time t_1 is chosen on the basis of the steady state of the non-linear system and the integrals (24) are determined by numerical calculations.

The formulas (25) represent the necessary conditions of optimization, and in practical applications the results received do not require the Hesse-Matrix computations.

To linearize the non-linear systems (1) we used the following basis functions:

– Taylor's series expansion around equilibrium point
– Taylor's series expansion around equilibrium point with the transient components.

In this case two different optimal equations were obtained. The optimal matrices A^*_1 and A^*_2 with the numerical values of elements taken from the example presented in section 2, are shown below (A^*_1 represents the case of Taylor's series expansion around equilibrium point, and A^*_2 represents the case of Taylor's series expansion around equilibrium point with the transient components).

$$A^*_1 = \begin{bmatrix} -1466.31 & -1919.04 & -20.5118 \\ 10 & -120.45 & -5 \\ 0 & 500 & -100 \end{bmatrix}$$

$$A^*_2 = \begin{bmatrix} -1457.07 & -2447.53 & -15.8207 \\ 10 & -120.45 & -5 \\ 0 & 500 & -100 \end{bmatrix}.$$

Matrix B^* is the same for both cases:

$$B^* = \begin{bmatrix} 1 \; 0 \; 0 \end{bmatrix}^T$$

The results of computations are presented in Figs. 9–14.

It is necessary to point out that the good results of the linearization of non-linear equation with the Taylor's expansion around equilibrium point with the transient components are obtained.

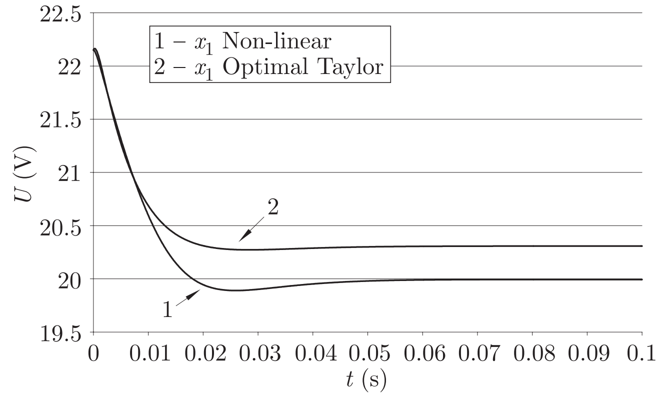

Fig. 9. The diagram of state variable x_1 – comparison between solution of non-linear equation (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point

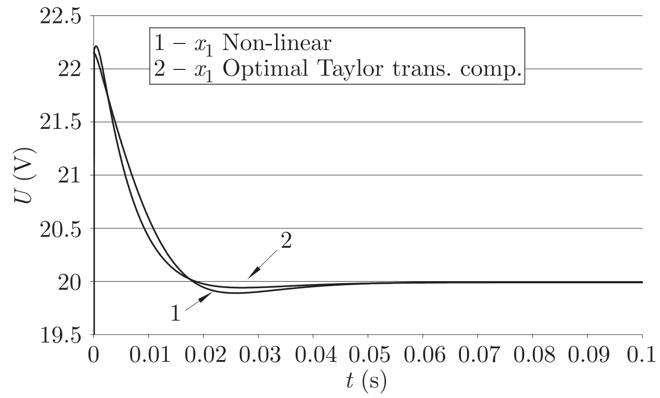

Fig. 10. The diagram of state variable x_1 – comparison between solution of non-linear Eq. (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point with the transient components

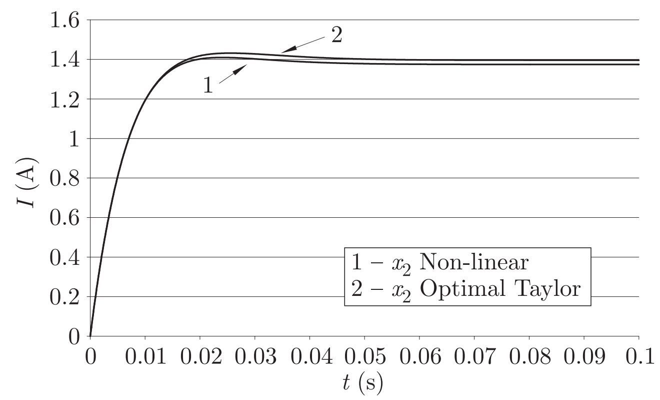

Fig. 11. The diagram of state variable x_2 – comparison between solution of non-linear Eq. (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point

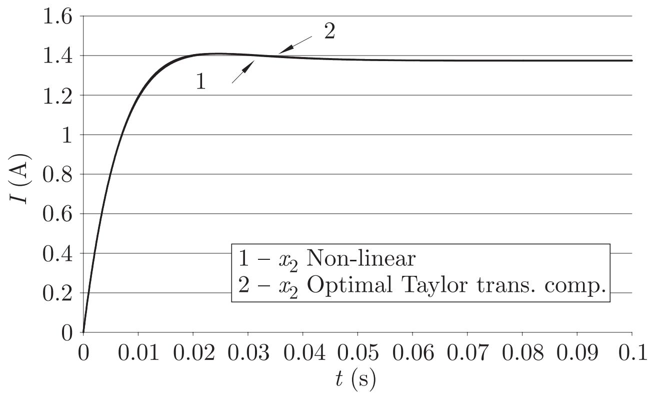

Fig. 12. The diagram of state variable x_2 – comparison between solution of non-linear Eq. (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point with the transient components

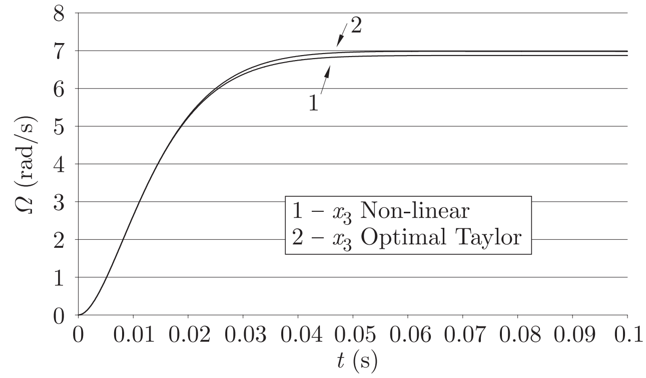

Fig. 13. The diagram of state variable x_3 – comparison between solution of non-linear Eq. (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point

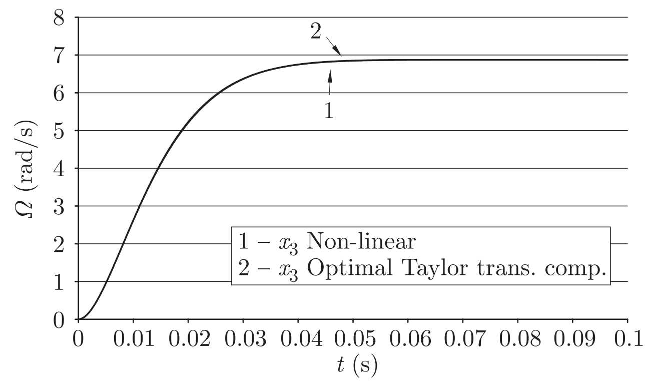

Fig. 14. The diagram of state variable x_3 – comparison between solution of non-linear Eq. (8) and solution of optimal Eq. (22) based on Taylor's series expansion around the equilibrium point with the transient components

5. Global linearization method

5.1. Variables transformation. In several cases, a non-linear equation (1) can be linearized by means of the state variables transformation that is defined using a global diffeomorphism [6–8,28]. Assuming that $f(x,u,t)$ is the continuous function and n-time differentiable, we apply the following variables transformation.

$$z = \phi(x) \tag{26}$$

where

$$\phi(x) = \begin{bmatrix} \Phi_1(x_1,\ x_2,\ ...,x_n) \\ \Phi_2(x_1,\ x_2,\ ...,x_n) \\ \dots\dots\dots\dots \\ \Phi_n(x_1,\ x_2,\ ...,x_n) \end{bmatrix}. \tag{27}$$

In this case Eq. (1) is transformed into the linear equation

$$\dot{z} = Az + Bv, \quad z(0) = \Phi\left[x(t=0)\right] \tag{28}$$

where v is a new input $v = u + f(x)$ and $f(x)$ is a non-linear combination of state variables $x_1, x_2, ..., x_n$. The solution of linear equation (28) is known:

$$z = e^{At}z(0) + \int_0^t e^{A(t-\tau)}Bv(\tau)d\tau. \tag{29}$$

Using the inverse transformation

$$x = \phi^{-1}(z) \tag{30}$$

we obtain vector $\tilde{x}(t)$ that satisfies relation $\tilde{x}(t) \cong x(t)$ over the whole state space where $t \to \infty$ and $x(t)$ is the solution of Eq. (1). The vector $\tilde{x}(t)$ results from the formula (30) and is computed using the iterative method which is presented in the next section.

5.2. Example of computations. We consider once more the same example presented in Figs. 1 and 2

$$\begin{aligned} \dot{x}_1 &= -a_1e^{ax_1} - a_2x_2 + u \\ \dot{x}_2 &= a_3x_1 - a_4x_2 - a_5x_3 \\ \dot{x}_3 &= a_6x_2 - a_7x_3 \\ x_1(0) &= V_{p,0},\ \ x_2(0) = 0,\ \ x_3(0) = 0 \end{aligned} \tag{31}$$

where the coefficients $a_1, ..., a_2$ are described in Section 2.

In order to linearize Eq. (31) the following transformation of variables is applied

$$\begin{aligned} z_1 &= x_3 \\ z_2 &= a_6x_2 - a_7x_3 \\ z_3 &= a_6\dot{x}_2 - a_7\dot{x}_3 = b_1x_1 - b_2x_2 - b_3x_3 \\ z_1(0) &= 0,\ \ z_2(0) = 0,\ \ z_3(0) = b_1V_{p,0} \end{aligned} \tag{32}$$

where:

$$b_1 = a_3a_6,\ \ b_2 = a_4a_6,\ \ b_3 = a_6a_5 - a_7^2 \tag{33}$$

on making basic transformations of Eq. (32) we obtain the system of linear differential equations

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 & -k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \tag{34}$$

or

$$\dot{z} = Az + Bv \tag{35}$$

where:

$$\begin{aligned} &v = b_1u - c_4e^{ax_1},\ \ c_4 = a_1a_3a_6,\ \ k_1 = -a_2a_3a_7 \\ &k_2 = -(a_2a_3 + a_4a_2 + a_6a_5),\ \ k_3 = a_4 + a_7. \end{aligned} \tag{36}$$

The required state variables $x(t)$ are determined by means of inverse transformation $x = \phi^{-1}(z)$. The inverse transformation can be presented as follows

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h_1z_1 + h_2z_2 + h_3z_3 \\ h_4z_4 + h_5z_2 \\ z_1 \end{bmatrix} \tag{37}$$

where:

$$h_1 = \frac{a_4a_7}{a_3a_6} + \frac{a_5}{a_3},\ \ h_2 = \frac{a_4 + a_7}{a_3a_6},$$

$$h_3 = \frac{1}{a_3a_6},\ \ h_4 = \frac{a_7}{a_6},\ \ h_5 = \frac{1}{a_6}.$$

Now we present a linear system, the analysis and solution of which are equivalent to the analysis and the solution of the non-linear system (31)

$$\dot{z} = Az + Bv, \quad z(0) = \phi(x_0) \tag{38}$$

$$v = b_1u + f(x) = b_1u - c_4e^{ax_1} \tag{39}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h_1z_1 + h_2z_2 + h_3z_3 \\ h_4z_1 + h_5z_2 \\ z_1 \end{bmatrix}. \tag{40}$$

To solve the system (38–40) we use the iterative method which is explained in the block diagram in Fig. 15.

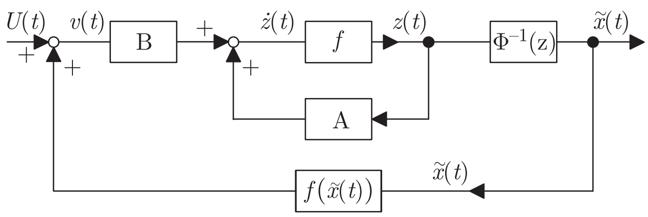

Fig. 15. Block diagram of linear system with the new input $v = u + f(\tilde{x})$

For the numerical solution the following recurrent model is applied

$$v_{i+1} = b_iu_i + f(\tilde{x}_i), \quad i = 0,1,2,...,N \tag{41}$$

$$z_{i+1} = Az_i + Bv_{i+1}, \quad i = 0,1,2,...,N \tag{42}$$

$$z_0 = \phi(x_0) \tag{43}$$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} h_1z_{1,i} + h_2z_{2,i} + h_3z_{3,i} \\ h_4z_{1,i} + h_5z_{5,i} \\ z_{1,i} \end{bmatrix} \quad i = 0,1,...,N \tag{44}$$

The iterative solution of equation

$$z_{i+1} = Az_i + Bv_{i+1} \tag{45}$$

is presented in Appendix 1.

Numerical analysis. For the calculations the parameter values (11) of the circuit shown in Fig. 1 are assumed. The difference between the numerical solution of Eq. (31) and that of Eqs. (41–44) can be characterized by the norm

$$\|x\|_{\max} = \max_{1\leqslant i\leqslant N} \|x_i\| \tag{46}$$

in this case

$$\begin{aligned} \|x_1(t) - \tilde{x}_1(t)\|_{\max} &\leqslant \varepsilon_1 \\ \|x_2(t) - \tilde{x}_2(t)\|_{\max} &\leqslant \varepsilon_2 \\ \|x_3(t) - \tilde{x}_3(t)\|_{\max} &\leqslant \varepsilon_3. \end{aligned} \tag{47}$$

Comparing the numerical solutions of Eq. (31) with Eqs. (41–44) we notice that the curves of state variables $x_1(t) = V_p(t)$, $x_2 = I_M(t)$ and $x_3 = \Omega(t)$ are identical. It results from small error values $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$, shown in Table 1, where h is the integration step in Runge-Kutta method. This method is also used to solve non-linear Eq. (31) as well as in the numerical implementation of the algorithm (41–44).

Table 1
The dependence of errors (42) on the value of the integration step h

h	ε_1	ε_2	ε_3
1.0 e-5	2.62705 e-02	1.5124 e-04	2.323 e-04
1.0 e-6	2.59553 e-03	1.517 e-05	2.336 e-05
1.0 e-7	2.5924 e-04	1.52 e-06	2.34 e-06
1.0 e-8	2.592 e-05	1.6 e-07	2.4 e-07

5.3. Generalization of a global linearization method. Let the

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the state vector, and $u \in R$ be a scalar function. We assume that the state equation can be presented as follows [5,8,29]:

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1) + x_2 + g_1(x,u) \\ \dot{x}_2 &= \phi_2(x_1,x_2) + x_3 + g_2(x,u) \\ \dot{x}_3 &= \phi_3(x_1,x_2,x_3) + g_3(x,u); \quad x(0) = x_0, \end{aligned} \tag{48}$$

where the functions ϕ_k and $g_k \in C_1$ for $k = 1,2,3$. To obtain the linear equation, we define the following change of variables

$$\begin{aligned} z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} &= \begin{bmatrix} x_1 \\ \phi_1(x_1) + x_2 \\ \phi(x_1,x_2) + x_3 \end{bmatrix} = \phi(x), \\ z\,(0) &= \phi\left[x\,(t=0)\right]. \end{aligned} \tag{49}$$

The inverse transformation that expresses the vector x in the function of vector z is the following

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 - \phi_1(z_1) \\ z_3 - \phi_2(z_1, z_2 - \phi_1(z_1)) \end{bmatrix} = \phi^{-1}(z). \tag{50}$$

Using (48) and (49), after necessary transformations, we have:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} g_1(x,u) \\ \bar{g}_2(x,u) \\ \bar{g}_3(x,u) \end{bmatrix}, \tag{51}$$

$$z\,(0) = \phi\left[x\,(t=0)\right] \tag{52}$$

or

$$\dot{z} = Az + g(x,u) \tag{53}$$

where:

$$g(x,u) = \begin{bmatrix} g_1(x,u) \\ \bar{g}_2(x,u) \\ \bar{g}_3(x,u) \end{bmatrix} \tag{54}$$

$$\bar{g}_2(x,u) = \frac{\partial \phi_1}{\partial x_1}\left[\phi_1(x_1) + x_2 + g_1(x,u)\right] + g_2(x,u) \tag{55}$$

$$\begin{aligned} \bar{g}_3(x,u) = {} & \frac{\partial \phi_2}{\partial x_1}\left[\phi_1(x_1) + x_2 + g_1(x,u)\right] \\ & + \frac{\partial \phi_2}{\partial x_2}\left[\phi_2(x_1,x_2) + x_3 + g_2(x,u)\right] \\ & + \phi_3(x_1,x_2,x_3) + g_3(x,u) \\ & + k_1 z_1 + k_2 z_2 + k_3 z_3. \end{aligned} \tag{56}$$

Parameters k_1, k_2 and k_3 are chosen in such a way as to ensure the stability of matrix A. These parameters allow us to analyse the linear circuit dynamics and by the same time the dynamics of the non-linear system. In practice the choice of the parameters k_1, k_2 and k_3 depends on the changes of the values of parameters of non-linear circuits, which are determined by the circuit structure and influence of some physical quantities e.g. temperature.

The numerical example. We consider once more the same example presented in Fig. 1 and described by Eq. (8). Having carried out some basic transformations, we obtain the following set of equations:

$$\begin{aligned} &\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a_2 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} g_1(x,u) \\ \bar{g}_2(x,u) \\ \bar{g}_3(x,u) \end{bmatrix} \\ &z_1(0) = V_p, \quad z_2(0) = -a_1 e^{aV_p}, \quad z_3(0) = a_3 V_p \end{aligned} \tag{57}$$

or

$$\dot{z} = Az + g(x,u) \tag{58}$$

where

$$g(x,u) = \begin{bmatrix} g_1(x,u) \\ \bar{g}_2(x,u) \\ \bar{g}_3(x,u) \end{bmatrix}. \tag{59}$$

In matrix A of Eq. (57) the parameter $a_2 (a_2 = 1/C)$ is introduced by means of transformation (49) in order to analyse the influence of capacity C on the dynamics of the non-linear circuit.

On the other hand, the transformation $z = \phi(x)$ can be built in such a way as to obtain the element of A, $a_{23} = 1$. However, this transformation lengthens the computation time due to a more complex form of $\bar{g}_2(x,u)$. In this case, we have

$$\begin{aligned} \phi_1(x_1) &= -a_1 e^{a x_1} \\ \phi_2(x_1,x_2) &= a_3 x_1 - a_4 x_2 \\ \phi_3(x_1,x_2,x_3) &= a_6 x_2 - a_7 x_3 \end{aligned} \tag{60}$$

$$
\begin{aligned}
\bar{g}_1(x,u) &= g_1(x,u) = u \\
\bar{g}_2(x,u) &= a a_1 e^{a x_1}(a_1 e^{a x_1} + a_2 x_2 - u) \\
&\quad - (1 + a_2)\cdot(a_3 x_1 - a_4 x_2 - a_5 x_3) \\
\bar{g}_3(x,u) &= a_6 x_1 - a_1 a_4 e^{a x_1} \\
&\quad - (a_2 a_4 + a_5 a_6) x_2 + a_5 a_7 x_3 + a_3 u.
\end{aligned} \tag{61}
$$

To analyse the influence of the electric circuit parameters on the circuit dynamics we assume the following form of k_1, k_2 and k_3 for matrix A:

$$
\begin{aligned}
k_1 &= a_6 = \frac{K_x}{J}, \quad k_2 = a_4 - a_3 = \frac{1}{L}(R_m - 1), \\
k_3 &= a_4 = \frac{R_m}{L}
\end{aligned} \tag{62}
$$

where J is the inertia moment, L and R_m denote the inductance and resistance of the rotor respectively, and K_x is the coefficient of the DC coil.

The eigenvalues of matrix A are investigated depending on the hypothetical changes of resistance R_m ($R_m = 12.045\ \Omega$ is the rated resistance). Assuming $R_{m.1} = 3\ \Omega$ and $R_{m.2} = 25\ \Omega$ we obtain the following eigenvalues:

$$
\begin{aligned}
R_{m.1} : \lambda_1 &= -25.08, \ \ \lambda_2 = -2.46 + j199.68, \\
\lambda_3 &= -2.46 - j199.68 \\
R_{m.2} : \lambda_1 &= -2.09, \ \ \lambda_2 = -123.96 + j681.26, \\
\lambda_3 &= -123.96 - j681.26.
\end{aligned}
$$

If we change the capacity C ($C = 500\ \mu\text{F}$ is the rated capacity) the eigenvalues are as follows:

$$
\begin{aligned}
C = 1000\ \mu\text{F} : \ \lambda_1 &= -4.45, \ \ \lambda_2 = -57.95 + j326.44, \\
\lambda_3 &= -57.95j - 326.44.
\end{aligned}
$$

For the $R_m = 12.045\ \Omega$ and the other rated parameters:

$$
\begin{aligned}
\lambda_1 &= -4.99, \ \ \lambda_2 = -57.73 + j443.84, \\
\lambda_3 &= -57.73 - j443.84.
\end{aligned}
$$

The diagrams showing $x_1(t) \cong \tilde{x}_1(t)$, $x_2(t) \cong \tilde{x}_2(t)$ and $x_3(t) \cong \tilde{x}_3(t)$ for the rated parameters and for $R_{m.1} = 3\ \Omega$ are presented in Fig. 16 and in Fig. 17.

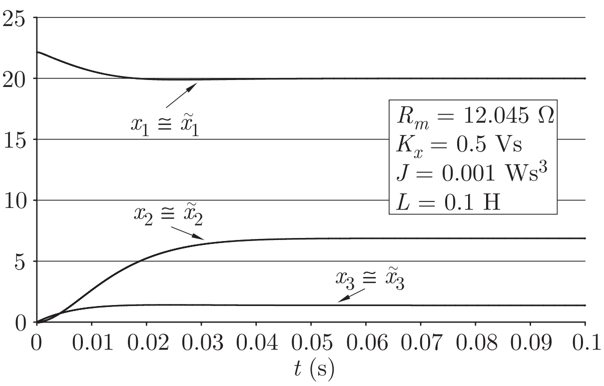

Fig. 16. The diagram presenting solution of linear equation with rated parameters and solution of linear equation with $R_{m.1} = 3\ \Omega$

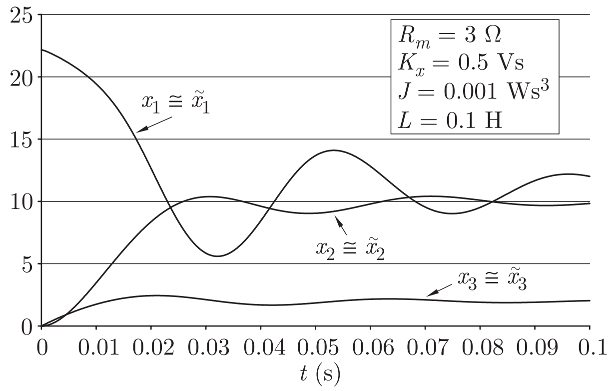

Fig. 17. The diagram presenting solution of linear equation with $R_{m.1} = 3\ \Omega$

5.4. Another example of the computations. Below we would like to show other applications of the global linearization method.

Analysis of the dynamic of asynchronous slip-ring motor. In this case we consider the following set of the non-linear equations [18]

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= -a_1 x_1 - a_5 x_2 + a_4 x_3 - b_4 x_2 x_5 - b_3 x_4 x_5 + e_1 \\
\frac{dx_2(t)}{dt} &= a_5 x_1 - a_1 x_2 + a_4 x_4 + b_4 x_1 x_5 + b_3 x_3 x_5 + e_2 \\
\frac{dx_3(t)}{dt} &= a_2 x_1 - a_3 x_3 - a_5 x_4 + b_2 x_2 x_5 + b_1 x_4 x_5 - e_3 \\
\frac{dx_4(t)}{dt} &= a_2 x_2 + a_5 x_3 - a_3 x_4 - b_2 x_1 x_5 - b_1 x_3 x_5 - e_4 \\
\frac{dx_5(t)}{dt} &= -c_2 x_5 + c_1 x_1 x_4 - c_1 x_2 x_3 - M
\end{aligned} \tag{63}
$$

where:

$x_1(t)$, $x_2(t)$ – the standard form of the stator current,
$x_3(t)$, $x_4(t)$ – the standard form of the rotor current,
$x_5(t)$ – the angular velocity.

Using the global linearization method we obtain the following linear equation:

$$
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-k_1 & -k_2 & -k_3 & -k_4 & -k_5
\end{bmatrix}
\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix}
+
\begin{bmatrix} g_1(x,u) \\ \bar{g}_2(x,u) \\ \bar{g}_3(x,u) \\ \bar{g}_4(x,u) \\ \bar{g}_5(x,u) \end{bmatrix} \tag{64}
$$

An inverse transformation:

$$
\begin{aligned}
x_1 &= z_1 \\
x_2 &= -\frac{a_1}{a_5} z_1 - \frac{1}{a_5} z_2 \\
x_3 &= -\frac{a_1^2 + a_5^2}{a_5} z_1 - \frac{a_1}{a_5} z_2 + z_3 \\
x_4 &= \frac{a_2 a_5 + a_3\left(a_1^2 + a_5^2\right)}{a_5^2} z_1 + \frac{a_1 a_3}{a_5^2} z_2 - \frac{a_3}{a_5} z_3 - \frac{1}{a_5} z_4 \\
x_5 &= \frac{a_1 a_2 a_5 + \left(a_1^2 + a_5^2\right) a_5^2 + a_2 a_3 a_5 + \left(a_1^2 + a_5^2\right) a_3^2}{a_5^2} z_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a_2a_5+a_1a_5^2+a_1a_3^2}{a_5^2}z_2 \\
&-\frac{a_3^2+a_5^2}{a_5}z_3-\frac{a_3}{a_5}z_4+z_5.
\end{aligned}
\tag{65}
$$

The solutions of non-linear and linear equations are presented in Figs. 18–20. We should note that the solutions of both equations are identical.

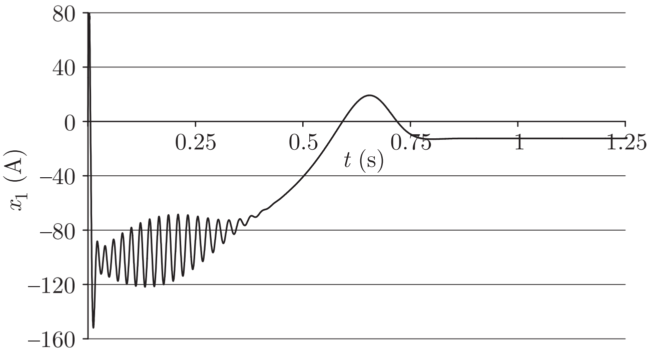

Fig. 18. The diagram of stator current

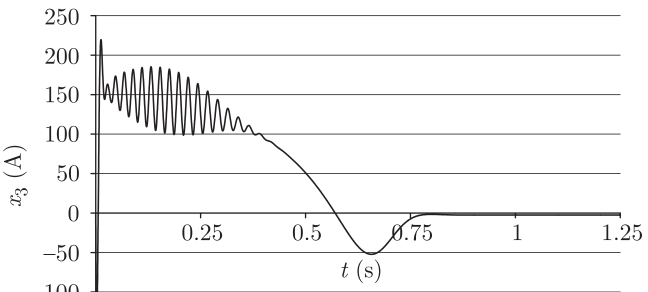

Fig. 19. The diagram of rotor current

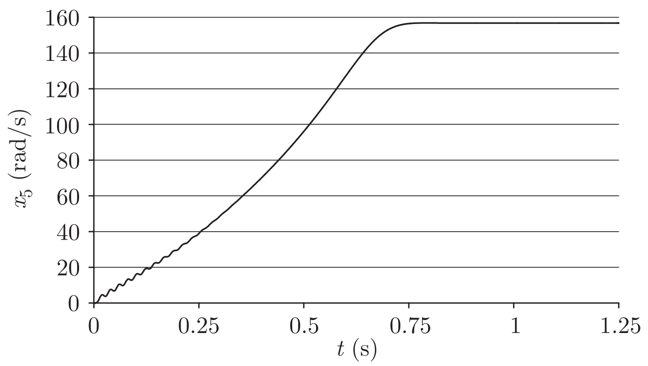

Fig. 20. The diagram of angular velocity

The electrical circuit with non-linear diode. The analysed non-linear electrical circuit containing a diode with non-linear characteristic is presented in Fig. 21 [28].

In the considered case the non-linear characteristic of the diode is as follows

$$i = av_c + bv_c^2 \tag{66}$$

Assuming

$$x_1 = v_c, \text{ and } x_2 = i_R \tag{67}$$

we have the following set of the non-linear equations

$$
\begin{gathered}
\dot{x}_1 = -a_1x_1 - a_2x_1^2 - a_3x_2 \\
\dot{x}_2 = a_4x_1 - a_5x_2, \\
x_1(0) = V_{c,0}, \;\; x_2(0) = 0.
\end{gathered}
\tag{68}
$$

Using the change of state variables defined as follows

$$
\begin{aligned}
z_1 &= x_2 \\
z_2 &= a_4x_1 - a_5x_2
\end{aligned}
\tag{69}
$$

and after some transformations, we obtain the linear equations

$$
\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k_1 & -k_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ v\end{bmatrix}
\tag{70}
$$

where:

$$
\begin{aligned}
&k_1 = a_1a_5 + a_3a_4, \;\; k_2 = a_1 + a_5 \\
&v = (-a_1 - 2a_2x_1)\dot{x}_1 - a_3\dot{x}_2 + k_1x_1 \\
&\quad + k_2(-a_1x_1 - a_2x_1^2 - a_3x_2).
\end{aligned}
\tag{71}
$$

The diagrams of $x_1(t) = v_c(t), \;\; x_2(t) = i_R(t)$ are shown in Figs. 22 and 23.

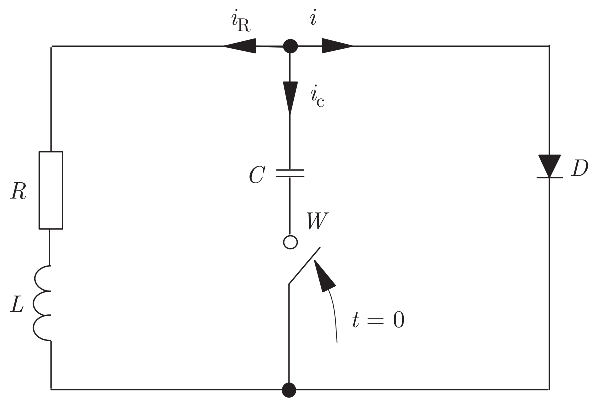

Fig. 21. Electric circuit with non-linear diode

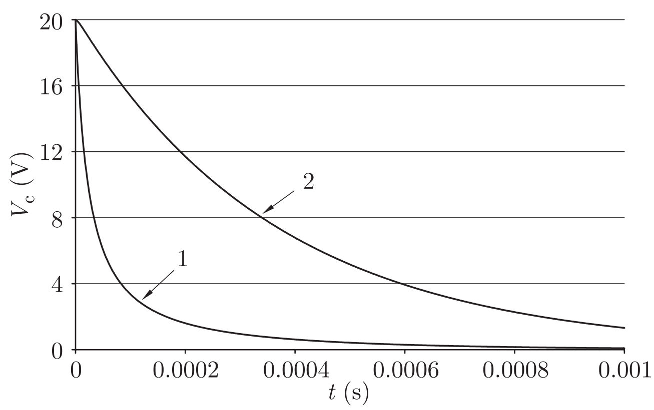

Fig. 22. The diagram of capacitor voltage

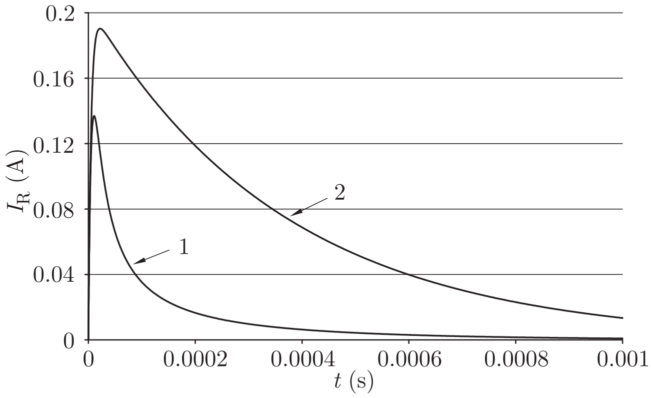

Fig. 23. The diagram of resistor current

In this case curve 1 shows both the non-linear solution and linear solution. Curve 2 shows the solution of the linear equation for $b = 0$.

6. Conclusions

In this paper several methods of the linearization of non-linear state equation have been presented. Some basic remarks concerning these methods can be made:

– the Taylor's series expansion assures a good approximation of non-linear equation for the small Δx deviations of the vector x. In the presented example, using Taylor's series expansion around equilibrium point with the transient components, a good approximation of the non-linear equation has been obtained;
– optimal linearization method assures a good approximation of the non-linear equation, however, it is expensive (time consuming);
– global linearization method assures convergence of linear solution with respect to non-linear solution with the norm maximum $[\tilde{x}(t) \cong x(t)]$.

In order to obtain suitable formalism of computations for the global linearization method, we use the following algorithm resulting from the example presented in section 5.2:

1) introduce non-linear functions $f(x, u, t)$ – the right-hand side of the non-linear state equation,
2) define and introduce functions: ϕ_1, ϕ_2, ϕ_3 and $\frac{\partial \phi_1}{\partial x_1}$, $\frac{\partial \phi_2}{\partial x_2}$, $\frac{\partial \phi_3}{\partial x_3}$,
3) introduce a direct and inverse change of variables: $z = \phi(x)$, $\tilde{x} = \phi^{-1}(z)$
4) define and introduce coefficients k_1, k_2, and k_3.

In the computations we apply the Runge-Kutta method of the 4$^{\mathrm{th}}$ order with the integration step $h = 10^{-6}$ s for the non-linear case and $h = 10^{-11}$ s for the linear case in the global linearization method.

The above method can be generalized for the n-dimensional space $(x \in R^n)$ [8].

Appendix 1

To solve Eq. (35) we can use the following iterative method

$$z\left[(k+1)T\right] = e^{AT} z(kT) + e^{A(k+1)T} \int_{kT}^{(k+1)T} e^{-A\tau} Bv(\tau) d\tau \tag{72}$$

where

$$kT \leqslant t \leqslant (k+1)T, k = 0, 1, 2, ... \tag{73}$$

Substituting the following relations into Eq. (72)

$$e^{AT} = \sum_{k=0}^{\infty} \frac{(AT)^k}{k!} \tag{74}$$

and

$$(e^{AT} - 1)A^{-1} = T \sum_{k=0}^{\infty} \frac{(AT)^k}{(k+1)!} \tag{75}$$

we obtain

$$z\left[(k+1)T\right] = \sum_{k=0}^{\infty} \frac{(AT)^k}{k!} z(kT) + T \sum_{k=0}^{\infty} \frac{(AT)^k}{(k+1)!} Bv(kT). \tag{76}$$

Using equations

$$A_1 = \sum_{k=0}^{\infty} \frac{(AT)^k}{k!} \text{ and } A_2 = T \sum_{k=0}^{\infty} \frac{(AT)^k}{(k+1)!} \tag{77}$$

we calculate the sums of the series (77) according to the convergence criterion:

$$\|S_{k+1}\| - \|S_k\| \leqslant \varepsilon, \text{ i.e } \varepsilon = 10^{-5} \tag{78}$$

$\|S\| = \max_j \sum_l a_{ij}$, where a_{ij} are elements of matrix A_1 and matrix A_2. Finally, we obtain the following recurrent equation

$$z(k+1) = A_1 z(k) + A_2 Bv(k). \tag{79}$$

Appendix 2

DEFINITION 1. A replacement of the non-linear system (1) by its linear approximation $\Delta\dot{x}(t) = A\Delta x(t) + B\Delta u(t)$ is called the "linearization by the Taylor's series expansion" of the non-linear system (1), where $A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_{eq} \\ u=u_{eq}}}$, $B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_{eq} \\ u=u_{eq}}}$ and the non-linear part $R = 0$.

DEFINITION 2. The linear equation obtained by neglecting the non-linear part R of Eq.(1) is called the linear approximation of the non-linear system.

DEFINITION 3. If the norm $\|x_{i,L}(t) - x_{L,NL}(t)\|_{\max}$ is less than prescribed value ε, i.e. $\|x_{i,L} - x_{i,NL}\|_{\max} < \varepsilon$ than the non-linear system is called weakly non-linear one, otherwise it is called strongly non-linear:
$x_{i,L} - i^{th}$ state variable of the linear system,
$x_{i,NL} - i^{th}$ state variable of the non-linear system.

DEFINITION 4. The system (1) is called BIBS (bounded-input bounded state) stable if for any bounded (norm) input u the state vector x is also (norm) bounded, i.e.

$$\|u\| < M \text{ implies } \|x\| < N \text{ for some finite numbers } M > 0 \text{ and } N > 0 \text{ where } \|\| \text{ denotes the norm of vector.} \tag{80}$$

THEOREM 3. (T. Kaczorek) [8]. The closed-loop nonlinear system is BIBS stable if the following conditions are satisfied.

1) There exists a global diffeomorphism such that (28) holds for $v = u + f(x)$. This diffeomorphism is defined as follows:

$$z = \phi(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \dots \\ \phi_n(x) \end{bmatrix} \tag{81}$$

with the following properties:
i) $\phi(x)$ is invertible, i.e. there exists a function $\phi^{-1}(z)$ such that

$$\phi^{-1}(\phi(x)) = x \text{ for all } x \text{ in } R^n \tag{82}$$

ii) $\phi(x)$ and $\phi^{-1}(z)$ are both smooth mappings (have continues partial derivatives of any order).

A given transformation (81) is a global diffeomorphism if it is a smooth function in R^n and the jacobian matrix

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \dots & \frac{\partial \phi_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial \phi_n}{\partial x_1} & \dots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix} \tag{83}$$

is non-singular for all x in R^n.

2) The function $f(x)$ is continuous and bounded for all x in R^n.

3) All eigenvalues of matrix A have negative real parts.

4) The function $x = \phi^{-1}(z)$ is bounded for all z in R^n and $t \in [0, +\infty]$.

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