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# On temperature distributions in a semi-infinite periodically stratified layer

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**Abstract.** The paper presents some problems of heat conduction in a semi-infinite periodically stratified layer. The layer is subjected to acting a constant temperature on the part of boundary, normal to the layering. The free heat exchange with surroundings is assumed on the remaining part of the boundary. The composite layer is supposed to be composed of $n$ periodically repeated two-component lamina. The problem is solved in two ways: ($1^0$) directly as a heat conduction problem, ($2^0$) by using model with microlocal parameters [1,2]. The main aim of the paper is a comparison of the obtained results and to conclude possibilities of applications of the homogenized model with microlocal parameters.

**Key words:** temperature distribution, heat conduction, semi-infinite periodically stratified layer.

## 1. Introduction

Periodically layered composite materials can be made by man (laminated composites) or can be found in nature (varved clays, sandstone-slates, sandstone-shales, thin-layered limenstones). The knowledge of temperature, heat flux distributions in laminated composites is very important in some problems of civil engineering, mechanical engineering and geophysics. Heat conduction problems of periodically layered composites can be solved by using "classical" descriptions. However, in the case of large number of repeated layers being components of composite the compliance of continuity conditions on interfaces leads to complicated problems for direct analytical and numerical approaches. For these reasons applications of some homogenized models seem to be useful. One of them is the homogenized model with microlocal parameters [1,2].

The model has been derived by using the nonstandard analysis combined with some a priori postulated assumptions. An application of the homogenization procedure leads to equations given in terms of unknown macrotemperature and thermal microlocal parameters. The microlocal parameters are to evaluate not only mean but also local values of heat fluxes in every material component of the composite. The homogenized model with microlocal parameters has been applied in many problems of periodically layered composites (see, for example for crack problems [3–6], inclusion problems [7–9]). The homogenization procedure with microlocal parameters was applied to the modelling of periodically layered inelastic composites [10,11]. The applications of the homogenized models are partially summarized in [12,13]. However, there are only few comparative results obtained within the framework of the homogenized model (approximated solutions) and an exact approach (exact solution).

This paper deals with the heat conduction problem for a semi-infinite periodically laminated layer. The repeated lamina is assumed to be composed of two isotropic homogeneous non-deformable layers. The perfect thermal contact between the layers is supposed. The part of boundary being a cross section of the laminated composite is kept at a constant temperature. On the remaining parts of boundary the free exchange condition with surroundings is assumed. The above problem is solved by using the classical heat conduction description as well as by an application of the homogenized model with microlocal parameters [1,2]. The main aim is to compare the obtained results within the framework of both approaches.

The paper stands for a continuation of our previous study [14,15] connected with the comparisons of the solutions to heat conduction problems in laminated composites obtained in the both above discussed ways. The paper [14] deals with the problem of temperature distribution in the semi-infinite laminated layer heated by a constant heat flux on the part of boundary normal to the layering. In the paper [15] some heat conduction problem in a laminated half-space is solved.

## 2. Formulations and solutions of the problem

Consider a rigid, semi-infinite layer composed of $n$ periodically repeated lamina (Fig. 1). Let $l_1, l_2$ denote the thicknesses of the subsequent layers, $a$ be the thickness of the fundamental layer. Let $\delta_1 = l_1/a$, $\delta_2 = l_2/a$, be the dimensionless thicknesses of the subsequent layers and $\delta = l/a$ be the dimensionless thickness of the lamina. Let ($x,\ y,\ z$) be dimensionless coordinates related to the thickness $a$. Thus, the periodically laminated body occupies the region $0 < x < 1$, $y > 0$, $-\infty < z < \infty$. Let $K_1, K_2$ denote the coefficients of heat conductivities of the subsequent layers of composite. The part of boundary being its cross-section $y = 0$ is kept at the constant temperature with the intensity $T_0$. On the surfaces $x = 0$ and $x = 1$ we consider the conditions of free exchange of heat with surroundings.

We will solve the determined above problem using two following approaches.

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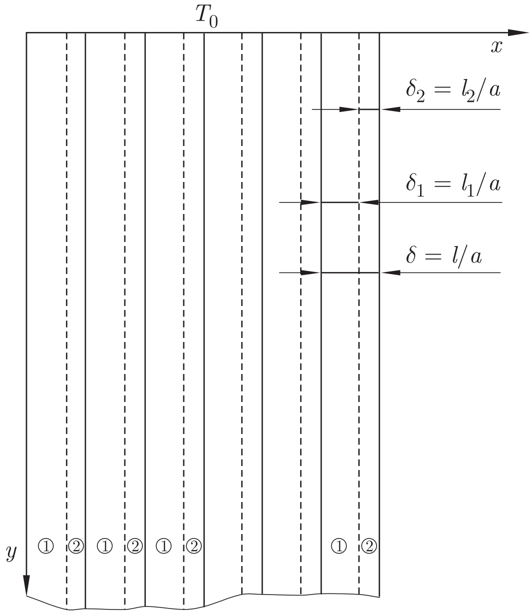


Fig. 1. The cross-section of scheme of the composite layer

**Approach 1. Formulation within the framework of equations of heat conduction.** Introduce the numbering $1, ..., 2n$ of the subsequent layers starting from the left side of the layer (from $x = 0$ to $x = 1$) (Fig. 1.). Let $T_i$, $i = 1, ..., 2n$ denote the temperature in the $i$-th layer. The equation of heat conduction in the $i$-th layer for the stationary case has the form

$$\partial^2 T_i/\partial x^2 + \partial^2 T_i/\partial y^2 = 0\,,\ i = 1, 2, ..., 2n. \qquad (1)$$

The boundary conditions are taken as follows

$$\begin{gathered} T_i = T_0, \text{ for } 0 < x < 1,\ y = 0,\ i = 1, ..., 2n, \\ \partial T_1/\partial x - Bi_1 T_1 = 0, \text{ for } x = 0,\ y > 0, \\ \partial T_{2n}/\partial x + Bi_2 T_{2n} = 0, \text{ for } x = 1,\ y > 0, \\ T_{2i-1} = T_{2i},\ K_1 \partial T_{2i-1}/\partial x = K_2 \partial T_{2i-1}/\partial x, \\ \text{for } x = (i-1)\delta + \delta_1,\ i = 1, ..., n, \\ T_{2i} = T_{2i+1},\ K_2 \partial T_{2i}/\partial x = K_1 \partial T_{2i+1}/\partial x \\ \text{for } x = i\delta,\ i = 1, ..., n-1, \end{gathered} \qquad (2)$$

where

$$Bi_1 = \alpha a/K_1, \quad Bi_2 = \alpha a/K_2 \qquad (3)$$

and $\alpha$ is the coefficient of heat exchange. Moreover, the regularity conditions in infinity

$$T_i \to 0, \text{ for } y \to \infty,\ i = 1, ..., 2n \qquad (4)$$

is considered.

The presented above problem will be solved by using Fourier sine transformation denoted in the case of function $f(x, y)$ as

$$\tilde{f}_{\mathrm{s}}(x, s) = F\left[f(x, y); y \to s\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x, y) \sin(sy)\, dy. \qquad (5)$$

The solution can be written in the form:

$$\begin{aligned} \widetilde{T}^{(\mathrm{s})}_{2i-1}(x, s) &= \sqrt{\frac{2}{\pi}} T_0 s^{-1} \\ &+ C_{4i-3}(s) \sinh(s((i-1)\delta + \delta_1 - x)) \\ &+ C_{4i-2}(s) \cosh(s((i-1)\delta + \delta_1 - x)), \\ &\quad i = 1, 2, ..., n \\ \widetilde{T}^{(\mathrm{s})}_{2i}(x, s) &= \sqrt{\frac{2}{\pi}} T_0 s^{-1} \\ &+ C_{4i-1}(s) \sinh(s(i\delta - x)) \\ &+ C_{4i}(s) \cosh(s(i\delta - x)),\ i = 1, 2, ..., n \end{aligned} \qquad (6)$$

where the functions $C_i(s)$ , $i = 1, 2, ..., 4n$ satisfy the following system of $4n$ linear algebraic equations:

$$\begin{aligned} & C_1(s)(s\cosh(s\delta_1) + Bi_1 \sinh(s\delta_1)) + C_2(s)(s\sinh(s\delta_1) \\ & \qquad + Bi_1 \cosh(s\delta_1)) = -\sqrt{\frac{2}{\pi}} T_0 Bi_1 s^{-1} \\ & C_{4i-2}(s) - C_{4i-1}(s) \sinh(\delta_2 s) \\ & \qquad - C_{4i}(s) \cosh(\delta_2 s) = 0, \quad i = 1, 2, ..., n \\ & \frac{K_1}{K_2} C_{4i-3}(s) - C_{4i-1}(s) \cosh(\delta_2 s) \\ & \qquad - C_{4i}(s) \sinh(\delta_2 s) = 0, \quad i = 1, 2, ..., n \\ & C_{4i}(s) - C_{4i+1}(s) \sinh(\delta_1 s) \\ & \qquad - C_{4i+2}(s) \cosh(\delta_1 s) = 0, \quad i = 1, 2, ..., n-1 \\ & \frac{K_2}{K_1} C_{4i-1}(s) - C_{4i+1}(s) \cosh(\delta_1 s) \\ & \qquad - C_{4i+2}(s) \sinh(\delta_1 s) = 0, \\ & \qquad i = 1, 2, ..., n-1 \\ & sC_{4n-1}(s) - Bi_2 C_{4n}(s) = \sqrt{\frac{2}{\pi}} T_0 Bi_2 s^{-1}. \end{aligned} \qquad (7)$$

The system of equations (7) is solved numerically.

**Approach 2. Formulation within the framework of the homogenized model with microlocal parameters.** If the number $n$ of repeated lamine is sufficiently large it seems to be suitable to use the homogenized models. One of them is the model with microlocal parameters given in [1,2]. The model in the stationary two-dimensional case of heat conduction problem is described by the following equation [3]:

$$\widetilde{K}^{-1} K^* \partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = 0 \qquad (8)$$

where $\theta$ is an unknown function interpreted as the macrotemperature and

$$\begin{gathered} \widetilde{K} = \eta K_1 + (1-\eta) K_2, \quad [K] = \eta(K_1 - K_2), \\ \widehat{K} = \eta K_1 + \frac{\eta^2}{1-\eta} K_2, K^* \equiv K_1 (1 - \frac{[K]}{\widehat{K}}) \\ = \frac{K_1 K_2}{(1-\eta) K_1 + \eta K_2},\ \ \eta = \frac{\delta_1}{\delta}. \end{gathered}$$

For small values of $\delta$ the following approximation for the temperature $T$ and its gradient can be written [3]:

$$T \approx \theta, \quad \partial T/\partial y \approx \partial\theta/\partial y,$$

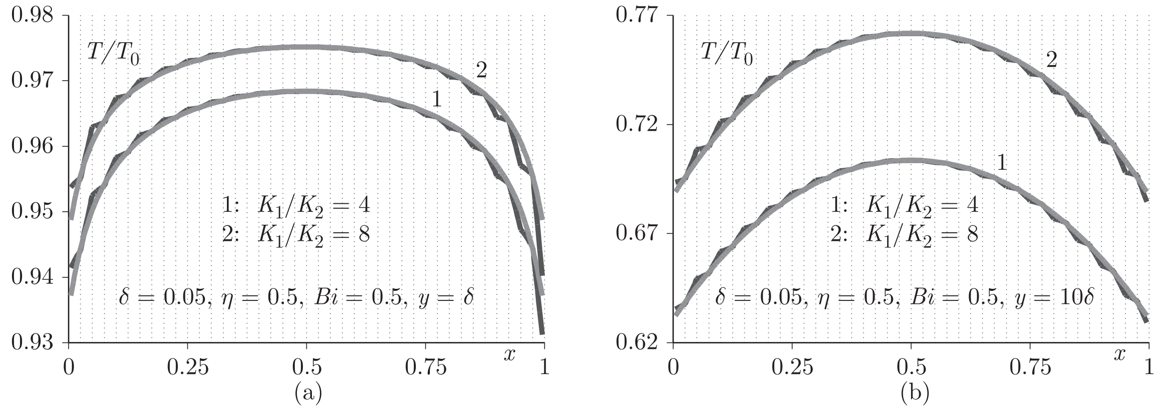


Fig. 2. The dimensionless temperature $T/T_0$ as functions of $x$ on two depths: (a) $y = \delta$ , (b) $y = 10\delta$

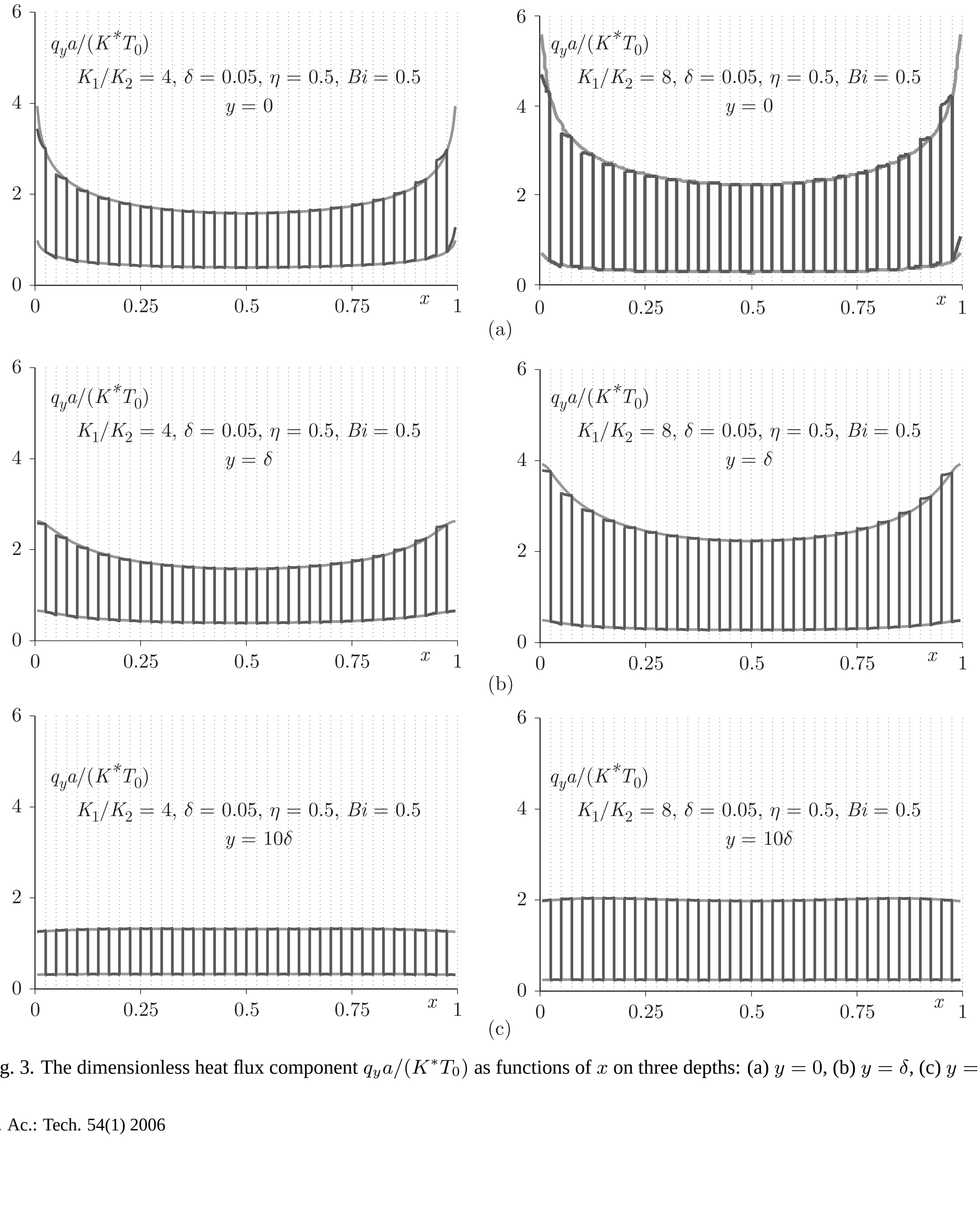


Fig. 3. The dimensionless heat flux component $q_y a/(K^* T_0)$ as functions of $x$ on three depths: (a) $y = 0$, (b) $y = \delta$, (c) $y = 10\delta$

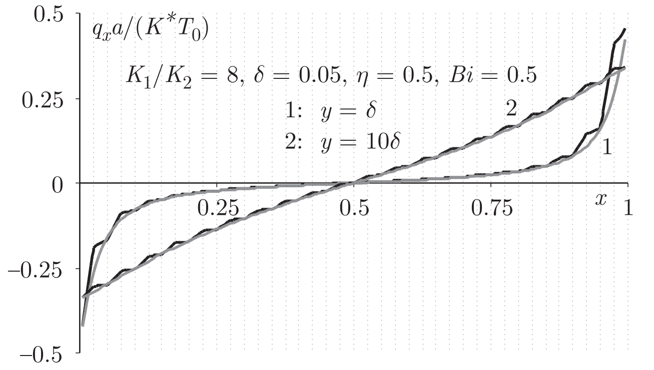


Fig. 4. The dimensionless heat flux component $q_x a/(K^* T_0)$ as functions of $x$ for two depths: $y = \delta$ and $y = 10\delta$

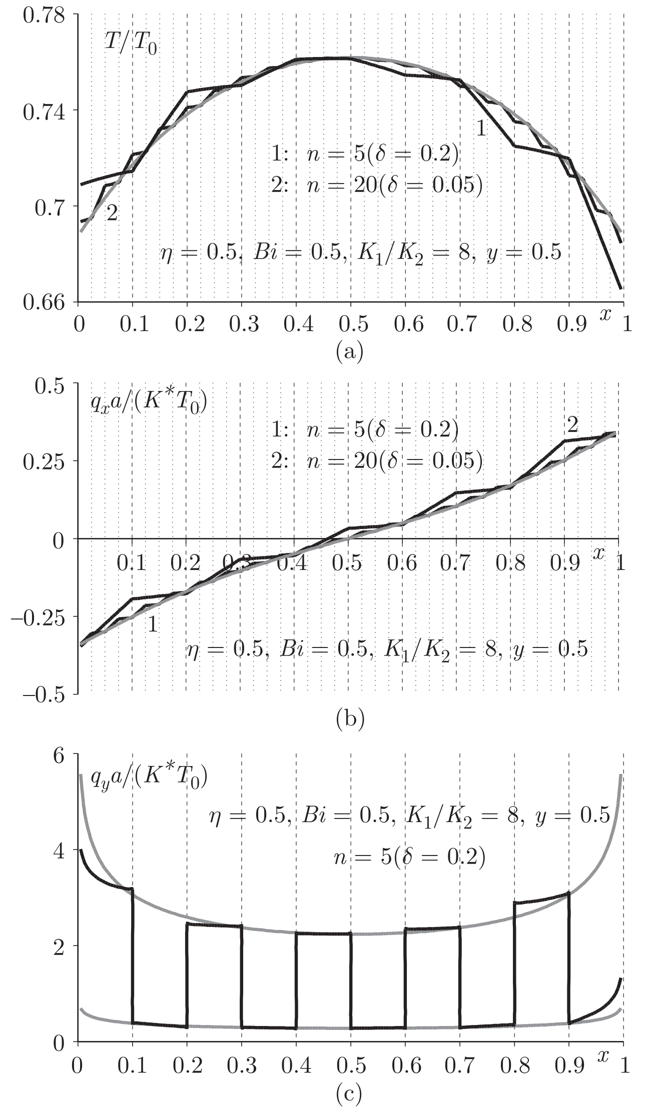


Fig. 5. The dimensionless temperature and heat fluxes

$$\partial T/\partial x = \begin{cases} (\partial\theta/\partial x)\,(K^*/K_1) & \text{for the layers of } 1-\text{st kind} \\ (\partial\theta/\partial x)\,(K^*/K_2) & \text{for the layers of } 2-\text{nd kind.} \end{cases} \quad (10)$$

The heat flux vector in a layer of the $i$-th kind, $i = 1, 2$, is given by

$$q^{(i)}(x,y) = (-K^*\partial\theta/\partial x, -K_i\partial\theta/\partial y,\ 0),\ \ i = 1,2. \quad (11)$$

The considered problem of semi-infinite composite layer is described by equation (8), and the boundary conditions:

$$\begin{aligned} &\theta = T_0, \ \text{for}\ 0 < x < 1,\ \ y = 0, \\ &\partial\theta/\partial x - Bi\theta = 0, \ \text{for}\ x = 0,\ \ y > 0; \\ &\partial\theta/\partial x + Bi\theta = 0, \ \text{for}\ x = 1,\ \ y > 0, \end{aligned} \quad (12)$$

the condition in infinity

$$\theta \to 0, \ \text{for}\ y \to \infty. \quad (13)$$

where

$$Bi = \alpha a/K^*. \quad (14)$$

The solution of the problem takes the form:

$$\tilde{\theta}_s\,(x,s) = \sqrt{\frac{2}{\pi}}\frac{T_0}{s} \times \{1 + Bi\,(A\,(S)\sinh(Sx) - B\,(S)\cosh(Sx))\} \quad (15)$$

where

$$A\,(S) = \frac{S\sinh(S) + Bi\cosh(S) - Bi}{(S^2 + Bi^2)\sinh(S) + 2BiS\cosh(S)}$$
$$B\,(S) = \frac{S\cosh(S) + Bi\sinh(S) + S}{(S^2 + Bi^2)\sinh(S) + 2BiS\cosh(S)}, \quad (16)$$
$$S = s\sqrt{\frac{\tilde{K}}{K^*}}.$$

The inverse Fourier transform of the function $\tilde{\theta}_s$ given in (15) will be calculated numerically and results will be shown in the form of graphs.

## 3. Numerical results

From the derived calculations there is seen that the finding of solution within the framework of the homogenized model with microlocal parameters is simpler than within the framework of the classical description (the continuity conditions on the interfaces are directly satisfied in the homogenized model). However, the homogenized model with microlocal parameters stands for some approximation of the classical approach.

So, the main aim is to compare the temperature and heat flux distributions given within the framework of the both approaches and to conclude about the applicability of the homogenized model. Moreover, the important question is how the results obtained within the homogenized model approximate the solutions given by classical formulation (Approach 1) together with decreasing of the lamina thickness $\delta$ (or with increasing of the number $n$ of fundamental layers being the components of the body).

The dimensionless temperature $T/T_0$ on the planes normal to the layering $y = \delta$; $y = 10\delta$ for two values of ratios of heat conductivities $K_1/K_2 = 4$, $K_1/K_2 = 8$, and $\eta = 0.5$, $Bi = 0.5$, $n = 20$, is shown in Fig. 2.

The waved curves present the solution given within the framework of classical description (Approach 1), the grey curves are given by the homogenized model (Approach 2). The presented curves in Fig. 2 show a well fitting of the approximated solutions with the obtained by the Approach 1.

The dimensionless heat fluxes $q_y a/(K^*T_0)$ directed parallel to the layering obtained within the both approaches are given in Fig. 3. The black curves present the component of heat flux for Approach 1, the grey curves are the solutions for Approach 2 and the upper curves show the components $q_y a/(K^*T_0)$ in layers of the first kind (with the coefficient of heat conductivity $K_1$), the lower curves in layers of the second kind (with the coefficient of heat conductivity $K_2$). These curves are extended to both kinds of layers for a better visibility.

Figure 4 shows the dimensionless heat flux component $q_x a/(K^*T_0)$ normal to the layering obtained within the framework of Approach 1 (the classical description) – black curves, and within the Approach 2 (the homogenized model) – grey curves.

The dimensionless temperature $T/T_0$ and the dimensionless heat flux $q_x a/(K^*T_0)$ as a function of $x$ on the depth $y = 0.5$ (the half of the composite layer thickness) are shown in Fig. 5a and Fig. 5b, respectively. The grey curves present the solutions obtained within the framework of the homogenized model, curves (1) are for $n = 5$ (repeated fundamental layers), curves (2) are for $n = 20$. Figure 5c shows the component of heat flux $q_y a/(K^*T_0)$ on the boundary plane $y = 0$. The grey curve represents the component for the homogenized model, the black curve for Approach 1 for $n = 5$. The curves in Fig. 5c are extended to both kinds of layers. From Figs. 5a-c it is seen that together with increasing of the number $n$ of fundamental layers being the composite components the differences between the solutions given for the homogenized model and the "exact" approach decrease. In the considered, problem the homogenized model stands for a good approximation of the results obtained within the framework of "classical" description (Approach 1).

## 4. Final remarks

In presented paper the problem of comparisons of the solutions obtained by using of the exact description with an assurance of the continuity conditions on interfaces and by employing of the homogenized model with microlocal parameters is considered on the example of the semi-infinite laminated two-component composite.

The choice of these considered shape of composites and boundary conditions is caused by possibilities of finding of solution for the exact approach. The homogenized model with microlocal parameters leads to the boundary value problems similar to ones for homogeneous anisotropic bodies.

The obtained results in the paper show that some differences between the solutions given for Approach 1 and 2 are for small depths from the boundary and they decrease together with increase of the distance from the body boundaries. The given above analysis confirms the conclusions presented in our previous papers [14,15], that the homogenized model with microlocal parameters can be applied for stationary problems of heat conduction in periodically laminated composites.

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