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Positive minimal realizations for singular discrete-time systems with delays in state and delays in control

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Abstract. The positive (minimal) realization problem for a class of singular discrete-time linear single-input, single-output systems with delays in state and delays in control is addressed. Solvability conditions for the positive (minimal) realization problem are established. It is shown that there exists a positive (minimal) realization of an improper transfer function $T(z) = n(z)/d(z)$ if the coefficients of polynomial $n(z)$ are non-negative and of the polynomial $d(z)$ are non-positive except the leading one, which should be positive. A procedure for computation of the positive (minimal) realization of the transfer function is proposed and illustrated by an example.

Key words: positive (minimal) realization, singular discrete-time systems, systems with delays, control.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in standard delay systems is given in [1] and in positive systems theory is given in the monographs [2,3]. Recent developments in positive systems theory and some new results are given in [4]. Realizations problem of positive linear systems without time-delays has been considered in many papers and books [2,3,5].

Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [6–9]. The realization problem for positive multivariable discrete-time systems with one time-delay was formulated and solved in [10,11].

The main purpose of this paper is to present a method for computation of positive (minimal) realization of an improper transfer function for a class singular discrete-time linear systems with delays in state and in control. It will be shown that there exists a positive (minimal) realization of improper transfer function if the coefficients of numerator polynomial are non-negative and of the denominator are non-positive (except the leading coefficient equal to 1).

To the best knowledge of the author the realization problem for singular linear systems with delays in the state vector and in control has not been considered yet.

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2. Preliminaries and problem formulation

Let $R_+^{n\times m}$ be the set of $n \times m$ matrices with entries from the field of real numbers and $R^n = Rn{\times}1$. The set of $n \times m$ matrices with real non-negative entries is denoted by $R_+^{n\times m}$ and $R_+^n = R_+^{n\times 1}$. The set of non-negative integers is denoted by Z_+ and the $n \times m$ identity matrix by I_n.

Consider the discrete-time linear system with one state delay and one input delay described by the equations

$$Ex(i+1) = A_0x(i)+A_1x(i-1)+B_0u(i)+B_1u(i-1) \quad (1a)$$

$$y(t) = cx(i) \quad i \in Z_+ \quad (1b)$$

where $x(i) \in R^n$, $u(i) \in R$, $y(i) \in R$ are the state vector, scalar input and scalar output respectively and $E,\ A_k \in R^{n\times n}$, $B_k \in R^n$, $k = 0, 1$, $c \in R^{1\times n}$.

It is assumed that det$E = 0$ and

$$\begin{aligned}&\det[Ez^2 - A_0z - A_1] \neq 0 \quad \text{for some}\\ &z \in C \quad \text{(the field of complex numbers)}\end{aligned} \quad (2)$$

The initial conditions for (1a) are given by

$$x(-i) \in R^n \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad u(-1) \in R \quad (3)$$

It is assumed that the initial conditions belong to the set X_0 of admissible initial conditions.

DEFINITION 1. The system (1) is called (internally) positive if for every $x(-k) \in R_+^n$, $k = 0, 1$, $u(-1) \in R_+$ and all inputs $u(i) \in R_+$, $i \in Z_+$ we have $x(i) \in R_+^n$ and $y(i) \in R_+$ for $i \in Z_+$.

Let us assume that the matrices $E,\ A_0,\ A_1,\ B_0,\ B_1,\ c$ have the following canonical forms [3].

$$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n\times n}, \quad A_0 = \begin{bmatrix} 0 & | & \bar{a}_0 \\ 0 & | & 0 \end{bmatrix} \in R^{n\times n},$$

$$\bar{a}_0 = \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{2n-3} \end{bmatrix} \in R^{n-1},$$

$$A_1 = \begin{bmatrix} \bar{A}_1 & | & \bar{a}_1 \\ \hline e_{n-1}^T & | & -a_{2(n-1)} \end{bmatrix} \in R^{n\times n},$$

$$\bar{A}_1 = \begin{bmatrix} 0 & | & 0 \\ \hline I_{n-2} & | & \end{bmatrix} \in R^{(n-1)\times(n-1)}, \quad \bar{a}_1 = \begin{bmatrix} a_0 \\ a_2 \\ \vdots \\ a_{2(n-2)} \end{bmatrix}, \tag{4}$$

$$B_0 = \begin{bmatrix} \bar{b}_0 \\ b_{0n} \end{bmatrix} \in R^n, \quad \bar{b}_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ \vdots \\ b_{0n-1} \end{bmatrix} \in R^{n-1},$$

$$B_1 = \begin{bmatrix} \bar{b}_1 \\ b_{1n} \end{bmatrix} \in R^n, \quad \bar{b}_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{1n-1} \end{bmatrix} \in R^{n-1},$$

$$c = e_n^T = [0 \ \dots \ 0 \ 1] \in R^{1\times n}.$$

THEOREM 1. The system (1) with (4) is positive if and only if

$$\bar{a}_k \in R_+^{n-1}, \ a_{2(n-1)} > 0 \quad \text{and} \quad B_k \in R_+^n, \ k = 0, 1 \tag{5}$$

P r o o f. *Sufficiency*. The equation (1a) for (4) can be written as

$$\begin{aligned} \bar{x}(i+1) = {} & \bar{a}_0 x_n(i) + \bar{A}_1 \bar{x}(i-1) + \bar{a}_1 x_n(i-1) \\ & + \bar{b}_0 u(i) + \bar{b}_1 u(i-1) \end{aligned} \tag{6a}$$

and

$$a_{2(n-1)} x_n(i-1) = e_n^T \bar{x}(i-1) + b_{on} u(i) + b_{1n} u(i-1) \tag{6b}$$

where

$$x(i) = \begin{bmatrix} \bar{x}(i) \\ x_n(i) \end{bmatrix} \in R^n, \ \bar{x}(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \\ \vdots \\ x_{n-1}(i) \end{bmatrix}, \ i \in Z_+.$$

If the conditions (5) are satisfied then using (6a) for $i = 0$ and the initial conditions (3) we may compute $\bar{x}(1) \in R_+$ and next from (6b) for $i = 1$ $x_n(1)$ and from (6a) $\bar{x}(2) \in R_+^{n-1}$. Continuing the procedure we may find $x(i) \in R_+^n$ for $i = 1, 2, \dots$ and from (1b) $y(i) = cx(i) \in R_+$ for $i = 1, 2, \dots$.

The necessity follows immediately from arbitrariness of the initial conditions (3) and of the input $u(i)$ and it can be shown in a similar way as for systems without delays [3].

Remark 1. Using (6b) we may eliminate $x_n(i)$ from (6a) and (1b) and we obtain

$$\begin{aligned} \bar{x}(i+1) = {} & \frac{\bar{a}_0 e_n^T}{a_{2(n-1)}} \bar{x}(i) + \left(\bar{A}_1 + \frac{\bar{a}_1 e_n^T}{a_{2(n-1)}} \right) \bar{x}(i-1) \\ & + \frac{\bar{a}_0 b_{0n}}{a_{2(n-1)}} u(i+1) + \left(\frac{\bar{a}_0 b_{1n} + \bar{a}_1 b_{0n}}{a_{2(n-1)}} + \bar{b}_0 \right) u(i) \\ & + \left(\frac{\bar{a}_1 b_{1n}}{a_{2(n-1)}} + \bar{b}_1 \right) u(i-1) \\ y(i) = {} & \frac{e_n^T}{2(n-1)} \bar{x}(i) + \frac{b_{0n}}{a_{2(n-1)}} u(i+1) + \frac{b_{1n}}{a_{2(n-1)}} u(i). \end{aligned}$$

The transfer function of (1) is given by

$$\begin{aligned} T(z) &= c[Ez - A_0 - A_1 z^{-1}]^{-1}(B_0 + B_1 z^{-1}) \\ &= c[Iz^2 - A_0 z - A_1]^{-1}(B_0 z + B_1). \end{aligned} \tag{7}$$

DEFINITION 2. Matrices (4) satisfying the condition (5) are called a positive realization of a given proper transfer function $T(z)$ if they satisfy the equality (7).

The realization is called minimal if the dimension $n \times n$ of $E, \ A_k, \ k = 0, 1$ is minimal among all realizations of $T(z)$.

The positive minimal realization problem can be stated as follows. Given an improper transfer function $T(z)$. Find a positive (minimal) realization of the $T(z)$.

Conditions for solvability of the positive (minimal) realization problem will be established and a procedure for computation of a positive (minimal) realization of $T(z)$ will be presented.

3. Problem solution

The transfer function (7) can be written in the form

$$T(z) = \frac{c(\mathrm{Adj}[Ez^2 - A_0 z + A_1])(B_0 z + B_1)}{\det[Ez^2 - A_0 + A_1]} = \frac{n(z)}{d(z)} \tag{8}$$

where

$$\begin{aligned} n(z) &= c(\mathrm{Adj}[Ez^2 - A_0 z - A_1])(B_0 z + B_1) \\ d(z) &= \det[Ez^2 - A_0 - A_1] \end{aligned} \tag{9}$$

and Adj stands for the adjoint matrix.

LEMMA 1. If the matrices E, A_0 and A_1 have the following forms (4) then

$$\begin{aligned} & \det[Ez^2 - A_0 z - A_1] \\ & = a_{a(n-1)} z^{a(n-1)} - a_{2n-3} z^{2n-3} - \dots - a_1 z + a_0. \end{aligned} \tag{10}$$

P r o o f. Expansion of the determinant with respect to the nth column yields

$$\begin{aligned} & \det[Ez^2 - A_0 z - A_1] \\ & = \begin{vmatrix} z^2 & 0 & 0 & \cdots & 0 & -a_1 z - a_0 \\ -1 & z^2 & 0 & \cdots & 0 & -a_3 z - a_2 \\ 0 & -1 & z^2 & \cdots & 0 & -a_5 z - a_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & z^2 & -a_{2n-3} z - a_{2(n-2)} \\ 0 & 0 & 0 & \cdots & -1 & a_{2(n-1)} \end{vmatrix} \\ & = a_{2(n-1)} z^{2(n-1)} - a_{2n-3} z^{2n-3} - \dots - a_1 z - a_0. \end{aligned}$$

LEMMA 2. If the matrices E, $A_0\,A_1$ have the forms (4) then the nth row $R_n(z)$ of the adjoint matrix Adj$[Ez^2-A_0z-A_1]$ has the form

$$R_n(z)=[1\;z^2\;\ldots\;z^{2(n-1)}]. \tag{11}$$

P r o o f. Taking into account that

$$(\mathrm{Adj}[Ez^2-A_0z-A_1])[Ez^2-A_0z-A_1]=I_nd(z)$$

it is easy to verify that

$$R_n(z)[Ez^2-A_0z-A_1]=[0\;\ldots\;0\;\;1]d(z).$$

Let a given improper transfer function have the form

$$T(z)=\frac{b_{2n-1}z^{2n-1}+\ldots+b_1z+b_0}{z^{2(n-1)}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0} \quad (b_{2n-1}\neq 0) \tag{12}$$

LEMMA 3. Let

$$T(z)=q_{-1}z+q_0+q_1z^{-1}+q_2z^{-2}+\ldots \tag{13}$$

Then

$$q_k\geq 0 \quad \text{for} \quad k=-1,0,1,\ldots \tag{14}$$

if

$$\begin{aligned}&a_i\geq 0 \quad \text{for} \quad i=0,1,\,\ldots,\,2n-3 \quad \text{and}\\ &b_j\geq 0 \quad \text{for} \quad j=0,\,1,\,\ldots,\,2n-1.\end{aligned} \tag{15}$$

P r o o f. From (12) and (13) we have

$$\begin{aligned}&b_{2n-1}z^{2n-1}+\ldots+b_1z+b_0\\ &=(z^{2n-1}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0)\\ &\times(q_{-1}z+q_0+q_1z^{-1}+q_2z^{-2}+\ldots).\end{aligned} \tag{16}$$

Equaling the coefficients at the same powers of z of (16) we obtain

$$\begin{aligned}&q_{-1}=b_{2n-1}>0,\\ &q_0=b_{2(n-1)}+a_{2n-3}b_{2n-1}>0,\;\ldots,\;q_{2(n-1)}\\ &=b_0+a_{2n-3}q_{2n-3}+\ldots+a_0q_0\geq 0,\;\ldots\end{aligned} \tag{17}$$

Knowing the coefficients of the polynomial we may find the matrices

$$A_0=\begin{bmatrix}0&\cdots&0&a_1\\0&\cdots&0&a_3\\\cdots&\cdots&\cdots&\cdots\\0&\cdots&0&a_{2n-3}\\0&\cdots&0&0\end{bmatrix},\quad A_1=\begin{bmatrix}0&0&\cdots&0&a_0\\1&0&\cdots&0&a_2\\0&1&\cdots&0&a_4\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&0&a_{2(n-2)}\\0&0&\cdots&1&-1\end{bmatrix} \tag{18}$$

such that

$$\begin{aligned}&\det[Ez^2-A_0z-A_1]\\ &=d(z)=z^{2(n-1)}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0.\end{aligned} \tag{19}$$

Using (8), (11) and (12) we obtain

$$\begin{aligned}T(z)&=c\frac{\mathrm{Adj}[Ez^2-A_0z-A_1]}{\det[Ez^2-A_0z-A_1]}(B_0z+B_1)\\ &=\frac{[1\;\;z^2\;\;\ldots\;\;z^{2(n-1)}]}{z^{2(n-1)}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0}\\ &\times\begin{bmatrix}b_{01}z+b_{11}\\b_{02}z+b_{12}\\\cdots\cdots\cdots\\b_{0n}z+b_{1n}\end{bmatrix}\\ &=\frac{b_{0n}z^{2n-1}+b_{1n}z^{2(n-1)}+\ldots+b_{01}z+b_{11}}{z^{2(n-1)}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0}\\ &=\frac{b_{2n-1}z^{2n-1}+\ldots+b_1z+b_0}{z^{2(n-1)}-a_{2n-3}z^{2n-3}-\ldots-a_1z-a_0}.\end{aligned} \tag{20}$$

Equaling the coefficients at the same powers of z of the numerators of (21) we obtain

$$b_{0n}=b_{2n-1},\;b_{1n}=b_{2(n-1)},\ldots,b_{01}=b_1,b_{11}=b_0. \tag{21}$$

THEOREM 2. There exists a positive minimal realization of (12) if

i) the coefficients $a_i\geq 0$ for $i=0,1,\ldots,2n-3$.

ii) the coefficients $a_j\geq 0$ for $j=0,1,\ldots,2n-1$.

P r o o f. If the condition ii) is satisfied then from (21) it follows that $B_0,\;B_1\in R_+^n$. If additionally the condition i) is satisfied then the conditions (5) hold and by Theorem 1 the realization is positive. Note that the dimension $n\times n$ of the matrices $A_0,\;A_1$ chosen of the forms (18) is minimal.

If the conditions of Theorem 2 are satisfied then a positive minimal realization of (12) can be found by the use of the following procedure.

PROCEDURE 1.

Step 1. Knowing the coefficients $a_i,i=0,1,\ldots,p-1$ of $d(z)$ find the matrices A_0 and A_1.

Step 2. Using (21) find the matrices B_0 and B_1.

Remark 2. The matrices E and c have the canonical forms (4) which are independent of $T(z)$.

Example 1. Given the transfer function

$$T(z)=\frac{2z^3+z^2+2z+1}{z^2-2z-3} \tag{22}$$

find its positive minimal realization.

It is easy to verify that the transfer function (22) satisfies the conditions of Theorem 2.

Using the Procedure 1 we obtain.

Step 1. From (22) we have and using (18) we obtain

$$A_0=\begin{bmatrix}0&a_1\\0&0\end{bmatrix}=\begin{bmatrix}0&2\\0&0\end{bmatrix},\;A_1=\begin{bmatrix}0&a_0\\1&-1\end{bmatrix}=\begin{bmatrix}0&3\\1&-1\end{bmatrix}. \tag{23}$$

Step 2. Using (23) we obtain in this case

$$b_0=\begin{bmatrix}b_{01}\\b_{02}\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},\;b_1=\begin{bmatrix}b_{11}\\b_{12}\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}. \tag{24}$$

The matrices E and c have the forms

$$E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\;c=\begin{bmatrix}0&1\end{bmatrix}. \tag{25}$$

The desired positive realization is given by (23), (24) and (25).

If the degree of denominator $d(z)$ of a given transfer function $T(z) = n(z)/d(z)$ is odd then multiplying the numerator and the denominator of $T(z)$ by z we obtain $T(z) = zn(z)/zd(z)$ with the denominator $zd(z)$ of even degree and we may apply the previous Procedure 1.

The obtained positive realization is not, in a general case, a minimal one.

Example 2. Find a positive realization of the transfer function

$$T(z) = \frac{2z^4 + 3z^3 + z^2 + 2z + 3}{z^3 - 2z^2 - z - 2} \tag{26}$$

Multiplying the numerator and the denominator of (26) by z we obtain

$$T(z) = \frac{2z^5 + 3z^4 + z^3 + 2z^2 + 3z}{z^4 - 2z^3 - z^2 - 2z} \tag{27}$$

The transfer function (27) satisfies the conditions of Theorem 2 and we may apply the Procedure 1.

$Step$ 1. From (27) we have $n = 3$ and using (18) we obtain

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \end{aligned} \tag{28}$$

$Step$ 2. Using (21) we obtain

$$B_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}. \tag{29}$$

The matrices E and c have the forms

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

The desired positive realization of (26) is given by (28), (29) and (30).

4. Multi delays systems

Consider the singular discrete-time linear system with two delays in state and two delays in input described by the equation

$$\begin{aligned} Ex(i+1) &= A_0 x(i) + A_1 x(i-1) + A_2 x(i-2) \\ &\quad + B_0 u(i) + B_1 u(i-1) + B_2 u(i-2) \end{aligned} \tag{31}$$

and (1b), where $x(i)$, $u(i)$ and $y(i)$ are defined in the same way as for (1a) and $A_2 \in R^{n\times n}$, $B_2 \in R^n$.

It is assumed that

$$\det[Ez^3 - A_0 z^2 - A_1 z - A_2] \neq 0 \quad \text{for some} \quad z \in C \tag{32}$$

The initial conditions for (31) are given by

$$\begin{aligned} &x(-i) \in R^n \quad \text{for} \quad i = 0, 1, 2 \quad \text{and} \\ &u(-j) \in R \quad \text{for} \quad j = 1, 2 \end{aligned} \tag{33}$$

and they belong to the set X_0 of admissible initial conditions.

It is also assumed that E and c have the canonical forms (4) and

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & \cdots & 0 & a_2 \\ 0 & \cdots & 0 & a_5 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{3n-4} \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & 0 & a_4 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{3n-5} \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{3(n-2)} \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, B_k = \begin{bmatrix} b_{k1} \\ b_{k2} \\ \vdots \\ b_{kn} \end{bmatrix}, k = 0, 1, 2. \end{aligned} \tag{34}$$

The following theorem can be shown in a similar way as Theorem 1.

THEOREM 3. The system described by (31), (1b) with matrices (4) and (34) is positive if and only if

$$\begin{aligned} &a_k \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, 3n-4 \quad \text{and} \\ &b_{ij} \geq 0 \quad \text{for} \quad i = 0, 1, 2; \quad j = 1, \ldots, n. \end{aligned} \tag{35}$$

LEMMA 4. If the matrices E, A_k, $k = 0, 1, 2$ have the canonical forms then

$$\begin{aligned} &\det[Ez^3 - A_0 z^2 - A_1 z - A_2] \\ &= z^{3(n-1)} - a_{3n-4} z^{3n-4} - \ldots - a_1 z - a_0. \end{aligned} \tag{36}$$

The proof is similar to the proof of Lemma 1.

LEMMA 5. If the matrices E, A_k, $k = 0, 1, 2$ have the canonical forms then the nth row $R_n(z)$ of the adjoint matrix Adj$[Ez^3 - A_0 z^2 - A_1 z - A_2]$ has the form

$$R_n(z) = \begin{bmatrix} 1 & z^3 & \cdots & z^{3(n-1)} \end{bmatrix}. \tag{37}$$

The proof is similar to the proof of Lemma 2.

Let a given improper transfer function have the form

$$\begin{aligned} T(z) &= \frac{n(z)}{d(z)} \\ &= \frac{b_{3n-1} z^{3n-1} + b_{3n-2} z^{3n-2} + \ldots + b_1 z + b_0}{z^{3(n-1)} - a_{3n-4} z^{3n-4} - \ldots - a_1 z - a_0}. \end{aligned} \tag{38}$$

Knowing $a_0, a_1, \ldots, a_{3n-4}$ of the denominator $d(z)$ we may find the matrices A_k, $k = 00, 1, 2$ of the forms (34) such that (36) holds. Using (36)–(38) we obtain

$$\begin{aligned} T(z) &= c[Ez - A_0 - A_1 z^{-1} - A_2 z^{-2}]^{-1} \\ &\quad \times (B_0 + B_1 z^{-1} + B_2 z^{-2}) \\ &= c[Ez^3 - A_0 z^2 - A_1 z - A_2]^{-1} (B_0 z^2 + B_1 z + B_2) \\ &= c \frac{\text{Adj}[Ez^3 - A_0 z^2 - A_1 z - A_0]}{\det[Ez^3 - A_0 z^2 - A_1 z - A_0]} \\ &\quad \times (B_0 z^2 + B_1 z + B_2) \\ &= \frac{\begin{bmatrix} 1 & z^3 & \ldots & z^{3(n-1)} \end{bmatrix}}{z^{3(n-1)} - a_{3n-4} z^{3n-4} - \ldots - a_1 z - a_0} \\ &\quad \times \begin{bmatrix} b_{01} z^2 + b_{11} z + b_{21} \\ b_{02} z^2 + b_{12} z + b_{22} \\ \cdots\cdots\cdots\cdots\cdots \\ b_{0n} z^2 + b_{1n} z + b_{2n} \end{bmatrix} = \end{aligned} \tag{39}$$

$$= \frac{b_{0n}z^{3n-1} + b_{1n}z^{3n-2} + b_{2n}z^{3(n-1)} + \ldots + b_{01}z^2 + b_{11}z + b_{21}}{z^{3(n-1)} - a_{3n-4}z^{3n-4} - \ldots - a_1 z - a_0}$$

$$= \frac{b_{3n-1}z^{3n-1} + b_{3n-2}z^{3n-2} + \ldots + b_1 z + b_0}{z^{3(n-1)} - a_{3n-4}z^{3n-4} - \ldots - a_1 z - a_0}.$$

Equalling the coefficients at the same powers of z of the numerators of (39) we obtain

$$\begin{aligned} &b_{0n} = b_{3n-1},\ b_{1n} = b_{3n-2}, \ldots, b_{01} = b_2, \\ &b_{11} = b_1,\ b_{21} = b_0. \end{aligned} \tag{40}$$

THEOREM 4. There exists a positive realization of (38) if the conditions (35) are satisfied. The proof is satisfied to the proof of Theorem 2.

If the conditions (35) are satisfied then a positive realization of (38) can be found by the use of the procedure similar to Procedure 1. The procedure is illustrated by the following example.

Example 3. Find a positive realization of the transfer function

$$T(z) = \frac{2z^5 + 3z^4 + 2z^3 + z^2 + z + 2}{z^3 - 2z^2 - 3z - 1} \tag{41}$$

It is easy to see that the transfer function (41) satisfies the conditions (35).

$Step$ 1. From (38) and (41) it follows that and using (34) we obtain

$$\begin{aligned} &A_0 = \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \\ &A_2 = \begin{bmatrix} 0 & a_0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned} \tag{42}$$

$Step$ 2. Using (41) and (41) we obtain

$$\begin{aligned} &B_0 = \begin{bmatrix} b_{01} \\ b_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\ &B_2 = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned} \tag{43}$$

The matrices E and c have the forms

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ c = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{44}$$

The desired positive realization of (41) is given by (42), (43) and (44).

Remark 3. If the degree of denominator $d(z)$ of a given transfer function $T(z) = n(z)/d(z)$ is equal to $3n-4$ $(3n-5)$ then multiplying the numerator and the denominator of $T(z)$ by $z(z^2)$ we obtain the desired transfer function and we may apply the proposed approach.

Let the transfer function, in general case, have the form

$$T(z) = \frac{n(z)}{d(z)},\ \deg n(z) = q,\ \deg d(z) = p. \tag{45}$$

Then the number of delays of the system is equal to

$$r = q - p. \tag{46}$$

If the matrices $E,\ A_k \in R^{n\times n},\ k = 0, 1, \ldots, r$, have the canonical forms then the minimal n is given by

$$n = \frac{q+1}{r+1}. \tag{47}$$

The formula (47) can be justified at follows.

If the matrix E has the canonical form then

$$(n-1)(r+1) = p \tag{48}$$

Taking into account (46) and solving (48) with respect to n we obtain the formula (47). In general case to find a positive realization of an improper transfer function of the form (45) we may use the following procedure.

PROCEDURE 2.

$Step$ 1. Knowing q and p and using (46) find the number r of delays of the system.

$Step$ 2. Knowing q and r and using (47) find the minimal n.

$Step$ 3. Knowing the coefficients a_k of the denominator find the matrices $A_k,\ k = 0, 1, \ldots, r$.

$Step$ 4. Using equalities similar to (40) find the matrices B_k for $k = 0, 1, \ldots, r$.

The Procedure 2 will be illustrated by the following example.

Example 4. Find a positive realization of the transfer function

$$T(z) = \frac{2z^7 + 3z^5 + 2z^3 + z^2 + z + 2}{z^5 - 4z^4 - 3z^3 - 2z^2 - z - 2}. \tag{49}$$

It is easy to see that the transfer function (49) satisfies the conditions (35).

To obtain the transfer function of the form (38) we multiply the numerator and the denominator of (49) by z and then we obtain

$$T(z) = \frac{2z^8 + 3z^6 + 2z^4 + z^3 + z^2 + 2z}{z^6 - 4z^5 - 3z^4 - 2z^3 - z^2 - 2z}. \tag{50}$$

Using Procedure 2 to (50) we obtain

$Step$ 1. Taking into account that in this case $q = 8$, $p = 6$ and using (46) we obtain

$$r = q - p = 2.$$

$Step$ 2. From (47) we have

$$n = \frac{q+1}{r+1} = 3.$$

$Step$ 3. Taking into account that $d(z) = z^6 - 4z^5 - 3z^4 - 2z^3 - z^2 - 2z$ and using (34) we obtain

$$\begin{aligned} &A_0 = \begin{bmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_5 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \\ &A_2 = \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_3 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}. \end{aligned} \tag{51}$$

$Step$ 4. In this case we have

$$\begin{bmatrix} 1 & z^3 & z^6 \end{bmatrix} \begin{bmatrix} b_{01}z^2 + b_{11}z + b_{21} \\ b_{02}z^2 + b_{12}z + b_{22} \\ b_{03}z^2 + b_{13}z + b_{23} \end{bmatrix} = 2z^8 + 3z^6 + 2z^4 + z^3 + z^2 + 2z \quad \text{and}$$

$$B_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}. \tag{52}$$

The matrices E and c have the forms

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{53}$$

The desired positive realization of (49) is given by (51), (52) and (53).

Remark 4. It is well known that the improper transfer function (45) can be always written as the sum

$$T(z) = T_{sp}(z) + p(z). \tag{54}$$

A positive realization of the strictly proper part $T_{sp}(z)$ can be found by the use of the method for standard systems [7,10,11] and the realization of the polynomial part

$$p(z) = p_r z^r + p_{r-1} z^{r-1} + \ldots + p_1 z + p_0 \tag{55}$$

there exists if and only if $p_i \geq 0$ for $i = 0, 1, \ldots, r$.

5. Concluding remarks

It has been shown that there exist a positive (minimal) realizations of the form (4) of improper transfer function (12) if the conditions of Theorem 2 are satisfied. Next the considerations have been extended for singular systems with many delays in state and in control. It has been shown that there exists a positive realization of any improper transfer function (45) of singular discrete-time linear system with matrices in canonical forms with r delays in state and r delays in control $(r = \deg n(z) - \deg d(z))$. Procedures for computation of a positive (minimal) realization of improper transfer function has been presented and illustrated by numerical examples. The presented method can be extended for a class of multi-input multi-output singular discrete-time and singular continuous-time linear systems with delays in state and in control.

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