BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 52, No. 4, 2004

On dual approach to piecewise-linear elasto-plasticity. Part II: Discrete models

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Abstract. The second part of the paper presents finite-dimensional models of linear elastic, elastic-strain hardening, elastic-perfectly plastic and rigid-perfectly plastic structures. These models can be seen as a result of discretisation procedure applied to the models of solids derived in the Part I. The implications of sub-dividing degrees of freedom into those with prescribed external forces and those with given displacements are discussed. It is pointed out that the dual energy principles given in this part of the paper can serve as a direct basis for numerical computations.

Keywords: imposed displacements, energy principles, mathematical programming.

1. Introduction

Models formulated at continuum level (Part I of the present paper) are valuable in understanding general nature of specific classes of problems in elasto-plasticity. However, they do not provide with the tools for solving problems that occur in engineering. It is necessary, therefore, to replace a general deformable solid by a proper discretised description of a truss, frame, plate or shell. The principles of discretisation are well established and an interested reader should consult numerous books on the Finite Element Method, the Finite Difference Method or the Boundary Element Method. Each of those methods allows the user to represent the state of the structure by a finite number of parameters, usually referred to as generalised variables. Differential equations are replaced by systems of algebraic equations and integrals are evaluated numerically as finite sums. In the sequel we show how such discretisation influences the dual principles derived in the Part I.

2. Notation and formulation

Let us consider a discrete model of a structure described by means of the following generalised variables: displacements $\boldsymbol{w} \in R^n$, strains $\boldsymbol{q} \in R^m$, loads $\boldsymbol{p} \in R^n$ and stresses $\boldsymbol{s} \in R^m$. In the sequel we call these single-column matrices "vectors" having in mind that they are elements of finite-dimensional vector spaces.

Generalised variables are properly defined if they preserve energy during the transition from the model at continuum level to the discrete model. Thus, there should be:

$$\int_V \boldsymbol{\varepsilon}^{\mathbf{T}} \boldsymbol{\sigma} dV = \boldsymbol{q}^{\mathbf{T}} \boldsymbol{s}$$
$$\int_S \mathbf{u}^{\mathbf{T}} \mathbf{t} dS = \boldsymbol{w}^{\mathbf{T}} \boldsymbol{p}. \tag{1}$$

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When these conditions are met, the kinematics of the discrete model is described by matrix equation

$$\boldsymbol{q} = \boldsymbol{C}\boldsymbol{w} \tag{2}$$

and equilibrium – by equation

$$\boldsymbol{p} = \boldsymbol{C}^{\mathbf{T}} \boldsymbol{s}. \tag{3}$$

In order to obtain well-defined discrete model, we have to assume that for each degree of freedom either a load is given, or a value of displacement is prescribed. Let the number of prescribed displacements be n_w and let the number of prescribed loads be n_p. Obviously, $n_w + n_p = n$. Now we are in a position to divide the vectors of displacements and loads into sub-vectors: $\boldsymbol{w} = \{\boldsymbol{w}_p, \boldsymbol{w}_w\}$ and $\boldsymbol{p} = \{\boldsymbol{p}_p, \boldsymbol{p}_w\}$. Sub-vectors $\boldsymbol{w}_p, \boldsymbol{p}_p \in R^{n_p}$ correspond to the degrees of freedom with given external forces, whereas sub-vectors $\boldsymbol{w}_w, \boldsymbol{p}_w \in R^{n_w}$ correspond to the degrees of freedom with given displacements.

Given such division of the degrees of freedom, the equation of kinematics becomes

$$\boldsymbol{q} = \boldsymbol{C}_p \boldsymbol{w}_p + \boldsymbol{C}_w \boldsymbol{w}_w \tag{4}$$

whereas equilibrium is governed by two equations

$$\boldsymbol{p}_{\mathbf{0}} = \boldsymbol{C}_p^{\mathbf{T}} \boldsymbol{s}, \quad \boldsymbol{r} = \boldsymbol{C}_w^{\mathbf{T}} s. \tag{5}$$

In equations (4) $\boldsymbol{C}_{\mathbf{p}}$ and $\boldsymbol{C}_{\mathbf{w}}$ are sub-matrices of the matrix of kinematics: $\boldsymbol{C} = [\boldsymbol{C}_{\mathbf{p}} | \boldsymbol{C}_{\mathbf{w}}]$.

By *static load* we understand given external forces p_0 applied to certain degrees of freedom of the structure, whereas by *kinematic load* we mean displacements w_0 enforced on the rest of degrees of freedom.

3. Linear-elastic structure

Elastic structures are governed by the constitutive law

$$\boldsymbol{s} = \boldsymbol{E}\boldsymbol{q} \tag{6}$$

or

$$\boldsymbol{q} = \boldsymbol{E}^{-1} \boldsymbol{s} \tag{7}$$

where $\boldsymbol{E}$ is an elasticity matrix.

Since linear-elastic analysis of a discretised structure is described by means of linear algebraic equations, the

Table 1
Linear elastic structure under static and kinematic load

Governing relations:

constitutive	$\boldsymbol{s} = \boldsymbol{E}\boldsymbol{q}$, $\boldsymbol{q} = \boldsymbol{E}^{-1}\boldsymbol{s}$
kinematics	$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{q}$
equilibrium	$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$
	$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{r}$
kinematic load	$\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{w}_{\mathbf{0}}$

Reduced system of governing equations:

	$\boldsymbol{w}_p$	$\boldsymbol{w}_w$	$\boldsymbol{s}$	$\boldsymbol{r}$	1	
$\nabla \boldsymbol{L}_{w_p} =$			$\boldsymbol{C}_p^T$		$-\boldsymbol{p}_0$	$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_{w_w} =$			$\boldsymbol{C}_w^T$	$-\boldsymbol{I}$		$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_{s} =$	$\boldsymbol{C}_p$	$\boldsymbol{C}_w$	$-\boldsymbol{E}^{-1}$			$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_{r} =$		$-\boldsymbol{I}$			$\boldsymbol{w}_0$	$= \boldsymbol{0}$

Potential:

$$L(\boldsymbol{w}_p, \boldsymbol{w}_w, \boldsymbol{s}, \boldsymbol{r}) = -\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{I}\boldsymbol{r} - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_{\mathbf{0}} + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_{\mathbf{0}}$$

Saddle point:

$$L(\boldsymbol{w}_{\mathbf{p}*}, \boldsymbol{w}_{\mathbf{w}*}, \boldsymbol{s}_*, \boldsymbol{r}_*) = \min_{\boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}} \max_{\boldsymbol{s}, \boldsymbol{r}} L(\boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r})$$

Primal problem:

$$\text{find} \min_{\boldsymbol{w}_p, \boldsymbol{w}_w, \boldsymbol{s}} \left\{ \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} - \boldsymbol{w}_p^{\mathbf{T}}\boldsymbol{p}_0 \right\}$$

subject to

$$\boldsymbol{C}_p\boldsymbol{w}_p + \boldsymbol{C}_w\boldsymbol{w}_w - \boldsymbol{E}^{-1}\boldsymbol{s} = \boldsymbol{0}$$
$$\boldsymbol{w}_w = \boldsymbol{w}_0$$

Dual problem:

$$\text{find} \max_{\boldsymbol{s}, \boldsymbol{r}} \left\{ -\frac{1}{2}\boldsymbol{s}^{T}\boldsymbol{E}^{-1}\boldsymbol{s} + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0 \right\}$$

subject to

$$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{r} = \boldsymbol{0}$$

proper template for converting it into a pair of dual constrained extremum problems is Table A1 of the Appendix to Part I. The derivation is summarised in Table 1. By eliminating strains from the complete set of governing equations, we obtain the reduced system with symmetric matrix of coefficients. Potential L is chosen in such a way that its derivatives with respect to displacements generate the left-hand side of the two first equations of the reduced system and its derivatives with respect to stresses and reactions yield the left-hand side of the two last equations. The saddle point problem corresponds to the free energy principle for displacements, stresses and reactions. The primal problem expresses the principle of minimum potential energy and the dual problem (after the sign of the cost function is changed) – the principle of minimum of the difference between the complementary energy and the work done by reactions on prescribed displacements.

The classical principles of minimum potential energy and minimum complementary energy are usually formulated in the presence of static load only. Taking $\boldsymbol{w}_{\mathbf{0}} = 0$ in Table 1, we obtain them as the following dual principles:

$$\text{find} \min_{\boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}} \left\{ \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_0 \right\}$$

subject to (8)

$$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}} - \boldsymbol{E}^{-1}\boldsymbol{s} = \boldsymbol{0}$$
$$\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{0}$$

$$\text{find } \max_{s,r}\left\{-\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}\right\}$$
subject to
$$\boldsymbol{C}_{\mathbf{P}}^{\mathbf{T}}\boldsymbol{s}=\boldsymbol{p}_{\mathbf{0}} \qquad (9)$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0}$$

Since reactions on fixed degrees of freedom do not contribute to the cost functions (they produce no work), the constraints on displacements can be omitted in the kinematic principle if we are not interested in obtaining reactions from the static principle. This leads to the well known principles:

$$\text{find } \min_{w,s}\left\{\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}-\boldsymbol{w}^{\mathbf{T}}\boldsymbol{p}_0\right\}$$
subject to
$$\boldsymbol{C}\boldsymbol{w}-\boldsymbol{E}^{-1}\boldsymbol{s}=\boldsymbol{0} \qquad (10)$$

$$\text{find } \max_{s}\left\{-\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}\right\}$$
subject to
$$\boldsymbol{C}^{\mathbf{T}}\boldsymbol{s}=\boldsymbol{p}_{\mathbf{0}}. \qquad (11)$$

Solving the constraint of (10) for strains and substituting result into the cost function, we obtain the unconstrained minimisation problem

$$\text{find } \min_{w}\left\{\frac{1}{2}\boldsymbol{w}^{\mathbf{T}}\boldsymbol{C}^{\mathbf{T}}\boldsymbol{E}\boldsymbol{C}\boldsymbol{w}-\boldsymbol{w}^{\mathbf{T}}\boldsymbol{p}_0\right\}. \qquad (12)$$

The gradient of this function vanishes if

$$\boldsymbol{K}\boldsymbol{w}=\boldsymbol{p}_{\mathbf{0}} \qquad (13)$$

where

$$\boldsymbol{K}=\boldsymbol{C}^{\mathbf{T}}\boldsymbol{E}\boldsymbol{C} \qquad (14)$$

is positive definite stiffness matrix. Thus, the kinematic principle (10) leads to the set of Eq. (13) of the displacement (direct stiffness) method. The constraint of the static principle (11) can be solved (not necessary uniquely) with respect to stresses. The resulting unconstrained minimisation problem yields the set of equations of the stress (direct flexibility) method.

If we are interested in purely kinematic loading, then we have to assume $\boldsymbol{p}_0=\boldsymbol{0}$ in Table 1. In that case the dual principles read:

$$\text{find } \min_{w_{\mathbf{P}},w_{\mathbf{w}},s}\left\{\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}\right\}$$
subject to
$$\boldsymbol{C}_{\mathbf{P}}\boldsymbol{w}_{\mathbf{P}}+\boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}}-\boldsymbol{E}^{-1}\boldsymbol{s}=\boldsymbol{0} \qquad (15)$$
$$\boldsymbol{w}_{\mathbf{w}}=\boldsymbol{w}_{\mathbf{0}}$$

$$\text{find } \max_{s,r}\left\{-\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_{\mathbf{0}}\right\}$$
subject to
$$\boldsymbol{C}_{\mathbf{P}}^{\mathbf{T}}\boldsymbol{s}=\boldsymbol{0} \qquad (16)$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0}.$$

Note that modifying this model for unilaterally imposed displacements is formally simple – unknown reactive forces become non-negative:

$$\text{find } \min_{w_{\mathbf{P}},w_{\mathbf{w}},s}\left\{\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}\right\}$$
subject to
$$\boldsymbol{C}_{\mathbf{P}}\boldsymbol{w}_{\mathbf{P}}+\boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}}-\boldsymbol{E}^{-1}\boldsymbol{s}=\boldsymbol{0} \qquad (17)$$
$$\boldsymbol{w}_{\mathbf{w}}\geqslant\boldsymbol{w}_{\mathbf{0}}$$

$$\text{find } \min_{s,r}\left\{-\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_{\mathbf{0}}\right\}$$
subject to
$$\boldsymbol{C}_{\mathbf{P}}^{\mathbf{T}}\boldsymbol{s}=\boldsymbol{0} \qquad (18)$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0},\ \boldsymbol{r}\geqslant\boldsymbol{0}.$$

As it has been already mentioned in the Part I of this paper, the response of structure to unilaterally imposed displacements is a LC-problem. It is not reducible to linear equations and the template given in Table A2 governs the derivation of dual principles (17)–(18).

4. Elastic-strain hardening solid

Let us consider a structure made of material that under uniaxial test expresses bi-linear behaviour, as shown in Fig. 2b of the Part I. We assume a piecewise-linear plastic potential

$$\boldsymbol{f}=\boldsymbol{H}\boldsymbol{\lambda}-\boldsymbol{N}^{T}\boldsymbol{s}+\boldsymbol{k}_0 \qquad (19)$$

where $\boldsymbol{H}$ is positive definite $(l\times l)$-matrix of strain hardening, vector $\boldsymbol{\lambda}$ contains l plastic multipliers, the columns of $(m\times l)$-matrix $\boldsymbol{N}$ are the gradients of yield surface and vector $\boldsymbol{k}_{\mathbf{0}}$ contains l prescribed plastic modulae.

The generalised strain consists now of elastic and plastic parts

$$\boldsymbol{q}=\boldsymbol{q}_{\boldsymbol{E}}+\boldsymbol{q}_{\mathbf{P}} \qquad (20)$$

where $\boldsymbol{q}_{\boldsymbol{E}}$ is defined by Eq. (7) and

$$\boldsymbol{q}_{\mathbf{P}}=\boldsymbol{N}\boldsymbol{\lambda} \qquad (21)$$

with additional constraints

$$\boldsymbol{f}\geqslant\boldsymbol{0},\quad \boldsymbol{\lambda}\geqslant\boldsymbol{0},\quad \boldsymbol{f}^{\mathbf{T}}\boldsymbol{\lambda}=0. \qquad (22)$$

Geometric interpretation of the constitutive relations is similar to that given for continuum in the Part I.

Using the derivation template given in Table A2 of the Appendix we obtain Table 2. For purely static loading the dual model reduces to the following principles:

Table 2
Elastic-strain hardening structure under static and kinematic load

Governing relations:

constitutive	$\boldsymbol{q} = \boldsymbol{q}_E + \boldsymbol{q}_p$
	$\boldsymbol{q}_{\boldsymbol{E}} = \boldsymbol{E}^{-1}\boldsymbol{s},\ \boldsymbol{q}_{\mathbf{P}} = \boldsymbol{N}\boldsymbol{\lambda}$
	$\boldsymbol{f} = \boldsymbol{H}\boldsymbol{\lambda} - \boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{k}_0$
	$\boldsymbol{f} \geqslant \boldsymbol{0},\ \boldsymbol{\lambda} \geqslant \boldsymbol{0},\ \boldsymbol{f}^{\mathbf{T}}\boldsymbol{\lambda} = \boldsymbol{0}$
kinematics	$\boldsymbol{C}_p\boldsymbol{w}_p + \boldsymbol{C}_w\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{q}$
equilibrium	$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0,\ \boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{r}$
kinematic load	$\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{w}_{\mathbf{0}}$

Reduced system:

	$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$	$\boldsymbol{w}_p$	$\boldsymbol{w}_w$	$\boldsymbol{S}$	$\boldsymbol{r}$	1	
$\nabla \boldsymbol{L}_{\boldsymbol{\lambda}} =$	$\boldsymbol{H}$			$-\boldsymbol{N}^{-T}$		$\boldsymbol{k}_0$	$\geqslant \boldsymbol{0}$
$\nabla \boldsymbol{L}_{w_p} =$				$\boldsymbol{C}_p^T$		$-\boldsymbol{p}_0$	$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_{w_w} =$				$\boldsymbol{C}_w^T$	$-\boldsymbol{I}$		$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_s =$	$-\boldsymbol{N}$	$\boldsymbol{C}_p$	$\boldsymbol{C}_w$	$-\boldsymbol{E}^{-1}$			$= \boldsymbol{0}$
$\nabla \boldsymbol{L}_r =$			$-\boldsymbol{I}$			$\boldsymbol{w}_0$	$= \boldsymbol{0}$

Potential:

$$\boldsymbol{L}(\boldsymbol{\lambda}, \boldsymbol{w}_p, \boldsymbol{w}_w, \boldsymbol{s}, \boldsymbol{r}) = \frac{1}{2}\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{H}\boldsymbol{\lambda} - \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} - \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{N}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{I}\boldsymbol{r} + \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{k}_0 - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_0 + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0$$

Saddle point:

$$L(\boldsymbol{\lambda}_*, \boldsymbol{w}_{\mathbf{p}*}, \boldsymbol{w}_{\mathbf{w}*}, \boldsymbol{s}_*, \boldsymbol{r}_*) = \min_{\boldsymbol{\lambda} \geqslant \boldsymbol{0}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}} \max_{\boldsymbol{s}, \boldsymbol{r}} L(\boldsymbol{\lambda}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r})$$

Primal problem:

find $\displaystyle\min_{\boldsymbol{\lambda}, \boldsymbol{w}_p, \boldsymbol{w}_w, \boldsymbol{s}} \left\{ \frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{H}\boldsymbol{\lambda} + \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} + \boldsymbol{\lambda}^T\boldsymbol{k}_0 - \boldsymbol{w}_p^{\mathbf{T}}\boldsymbol{p}_0 \right\}$

subject to

$$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}} - \boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} - \boldsymbol{N}\boldsymbol{\lambda} = \boldsymbol{0}$$
$$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$$
$$\boldsymbol{w}_w = \boldsymbol{w}_0$$

Dual problem:

find $\displaystyle\max_{\boldsymbol{\lambda}, \boldsymbol{s}, \boldsymbol{r}} \left\{ -\frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{H}\boldsymbol{\lambda} - \frac{1}{2}\boldsymbol{s}^T\boldsymbol{E}^{-1}\boldsymbol{s} + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0 \right\}$

subject to

$$-\boldsymbol{H}\boldsymbol{\lambda} + \boldsymbol{N}^T\boldsymbol{s} \leqslant \boldsymbol{k}_0$$
$$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{r} = \boldsymbol{0}$$

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find $\displaystyle\min_{\boldsymbol{\lambda}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{s}} \left\{ \frac{1}{2}\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{H}\boldsymbol{\lambda} + \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-1}\boldsymbol{s} + \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{k}_0 - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_0 \right\}$

subject to (23)

$$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} - \boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} - \boldsymbol{N}\boldsymbol{\lambda} = \boldsymbol{0}$$
$$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$$

find $\displaystyle\max_{\boldsymbol{\lambda}, \boldsymbol{s}} \left\{ -\frac{1}{2}\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{H}\boldsymbol{\lambda} - \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} \right\}$

subject to (24)

$$-\boldsymbol{H}\boldsymbol{\lambda} + \boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} \leqslant \boldsymbol{k}_0$$
$$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$$

For the sake of brevity, we omitted fixed degrees of freedom and reactions.

When the loading is purely kinematic, then the dual model reduces to:

$$\begin{aligned}&\text{find } \min_{\boldsymbol{\lambda},\boldsymbol{w}_\mathrm{p},\boldsymbol{w}_\mathrm{w},\boldsymbol{s}} \left\{\frac{1}{2}\boldsymbol{\lambda}^\mathbf{T}\boldsymbol{H}\boldsymbol{\lambda}+\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{\lambda}^\mathbf{T}\boldsymbol{k}_0\right\}\\ &\text{subject to}\\ &\boldsymbol{C}_\mathrm{p}\boldsymbol{w}_\mathrm{p}+\boldsymbol{C}_\mathrm{w}\boldsymbol{w}_\mathrm{w}-\boldsymbol{E}^{-1}\boldsymbol{s}-\boldsymbol{N}\boldsymbol{\lambda}=\boldsymbol{0}\\ &\boldsymbol{\lambda}\geqslant\boldsymbol{0}\\ &\boldsymbol{w}_\mathrm{w}=\boldsymbol{w}_0\end{aligned} \tag{25}$$

$$\begin{aligned}&\text{find } \max_{\boldsymbol{\lambda},\boldsymbol{s},\boldsymbol{r}} \left\{-\frac{1}{2}\boldsymbol{\lambda}^\mathbf{T}\boldsymbol{H}\boldsymbol{\lambda}-\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{r}^\mathbf{T}\boldsymbol{w}_0\right\}\\ &\text{subject to}\\ &-\boldsymbol{H}\boldsymbol{\lambda}+\boldsymbol{N}^\mathbf{T}\boldsymbol{s}\leqslant\boldsymbol{k}_0\\ &\boldsymbol{C}_\mathrm{p}^\mathbf{T}\boldsymbol{s}=\boldsymbol{0}\\ &\boldsymbol{C}_\mathrm{w}^\mathbf{T}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0}.\end{aligned} \tag{26}$$

All dual principles derived for elastic-strain hardening structures fall into category of convex QP-problems. Therefore, the response of such structure exists for any loading and is unique.

5. Elastic-perfectly plastic solid

Taking $\boldsymbol{H}=\boldsymbol{0}$ in Table 2, we obtain perfectly plastic structure. The result is given in Table 3. All remarks concerning uniqueness and existence of solutions made for continuum in the Part I, remain valid for the discretised structure.

For purely static loading the dual variational principles given in Table 3 reduce to:

$$\begin{aligned}&\text{find } \min_{\boldsymbol{\lambda},\boldsymbol{w}_\mathrm{p},\boldsymbol{s}} \left\{\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{\lambda}^\mathbf{T}\boldsymbol{k}_0-\boldsymbol{w}_\mathrm{p}^\mathbf{T}\boldsymbol{p}_0\right\}\\ &\text{subject to}\\ &\boldsymbol{C}_\mathrm{p}\boldsymbol{w}_\mathrm{p}-\boldsymbol{E}^{-1}\boldsymbol{s}-\boldsymbol{N}\boldsymbol{\lambda}=\boldsymbol{0}\\ &\boldsymbol{\lambda}\geqslant\boldsymbol{0}\end{aligned} \tag{27}$$

$$\begin{aligned}&\text{find } \max_{\boldsymbol{s}} \left\{-\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}\right\}\\ &\text{subject to}\\ &\boldsymbol{N}^\mathbf{T}\boldsymbol{s}\leqslant\boldsymbol{k}_0\\ &\boldsymbol{C}_\mathrm{p}^\mathbf{T}\boldsymbol{s}=\boldsymbol{p}_0\end{aligned} \tag{28}$$

and for purely kinematic loading they read as follows:

$$\begin{aligned}&\text{find } \min_{\boldsymbol{\lambda},\boldsymbol{w}_\mathrm{p},\boldsymbol{w}_\mathrm{w},\boldsymbol{s}} \left\{\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{\lambda}^\mathbf{T}\boldsymbol{k}_0\right\}\\ &\text{subject to}\\ &\boldsymbol{C}_\mathrm{p}\boldsymbol{w}_\mathrm{p}+\boldsymbol{C}_\mathrm{w}\boldsymbol{w}_\mathrm{w}-\boldsymbol{E}^{-1}\boldsymbol{s}-\boldsymbol{N}\boldsymbol{\lambda}=\boldsymbol{0}\\ &\boldsymbol{\lambda}\geqslant\boldsymbol{0}\\ &\boldsymbol{w}_\mathrm{w}=\boldsymbol{w}_0\end{aligned} \tag{29}$$

$$\begin{aligned}&\text{find } \max_{\boldsymbol{s},\boldsymbol{r}} \left\{-\frac{1}{2}\boldsymbol{s}^\mathbf{T}\boldsymbol{E}^{-1}\boldsymbol{s}+\boldsymbol{r}^\mathbf{T}\boldsymbol{w}_0\right\}\\ &\text{subject to}\\ &\boldsymbol{N}^\mathbf{T}\boldsymbol{s}\leqslant\boldsymbol{k}_0\\ &\boldsymbol{C}_\mathrm{p}^\mathbf{T}\boldsymbol{s}=\boldsymbol{0}\\ &\boldsymbol{C}_\mathrm{w}^\mathbf{T}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0}.\end{aligned} \tag{30}$$

6. Rigid-perfectly plastic solid

Substituting $\boldsymbol{E}^{-1}=\boldsymbol{0}$ into Table 3, we obtain dual principles for a structure made of rigid-perfectly plastic material (Table 4). They belong to the class of LP-problems which allows us to use very efficient simplex algorithm for solving them.

Keeping the prescribed static load $\boldsymbol{p_0}$ is inconvenient for rigid-perfectly plastic structure because such load may lead to contradictory constraints in the static principle. Hence, we take $\boldsymbol{p_0}=\boldsymbol{0}$ and obtain the following dual principles:

$$\begin{aligned}&\text{find } \min_{\dot{\boldsymbol{\lambda}},\dot{\mathbf{w}}_\mathrm{w},\boldsymbol{s}} \{\dot{\boldsymbol{\lambda}}^\mathbf{T}\boldsymbol{k}_0\}\\ &\text{subject to}\\ &\boldsymbol{C}_\mathrm{w}\dot{\boldsymbol{w}}_\mathrm{w}-\boldsymbol{N}\dot{\boldsymbol{\lambda}}=\boldsymbol{0}\\ &\dot{\boldsymbol{\lambda}}\geqslant\boldsymbol{0}\\ &\dot{\boldsymbol{w}}_\mathrm{w}=\dot{\boldsymbol{w}}_0\end{aligned} \tag{31}$$

$$\begin{aligned}&\text{find } \max_{\boldsymbol{r},\boldsymbol{s}} \{\boldsymbol{r}^\mathbf{T}\dot{\boldsymbol{w}}_0\}\\ &\text{subject to}\\ &\boldsymbol{N}^\mathbf{T}\boldsymbol{s}\leqslant\boldsymbol{k}_0\\ &\boldsymbol{C}_\mathrm{w}^\mathbf{T}\boldsymbol{s}-\boldsymbol{r}=\boldsymbol{0}.\end{aligned} \tag{32}$$

As already mentioned for continuum, this model can be interpreted as the optimisation of ultimate load. If the entries of $\boldsymbol{r}$ are treated as free variables, then we must completely define displacement rates $\dot{\boldsymbol{w}}_\mathbf{0}$. This requirement is relaxed, when certain information is available about the desired distribution of reactions. For example, vector $\boldsymbol{r}$ may be linearly dependent on vector $\hat{\boldsymbol{r}}$ of lower dimension: $\hat{\boldsymbol{r}}\in R^{\hat{n}}$ where $\hat{n}<n$. Then

$$\boldsymbol{r}=\boldsymbol{B}^\mathbf{T}\hat{\boldsymbol{r}} \tag{33}$$

where $(n\times\hat{n})$-matrix $\boldsymbol{B}$ is given. For such case model (31)–(32) has to be replaced by:

$$\begin{aligned}&\text{find } \min_{\dot{\boldsymbol{\lambda}},\dot{\boldsymbol{w}}_\mathrm{w}} \{\dot{\boldsymbol{\lambda}}^\mathbf{T}\boldsymbol{k}_0\}\\ &\text{subject to}\\ &\boldsymbol{C}_\mathrm{w}\dot{\boldsymbol{w}}_\mathrm{w}-\boldsymbol{N}\dot{\boldsymbol{\lambda}}=\boldsymbol{0}\\ &\dot{\boldsymbol{\lambda}}\geqslant\boldsymbol{0}\\ &\boldsymbol{B}\dot{\boldsymbol{w}}_\mathrm{w}=\dot{\boldsymbol{w}}_0\end{aligned} \tag{34}$$

Table 3
Elastic-perfectly plastic structure under static and kinematic load

Governing relations:

constitutive	$\boldsymbol{q} = \boldsymbol{q}_E + \boldsymbol{q}_P$
	$\boldsymbol{q}_E = \boldsymbol{E}^{-1}\boldsymbol{s},\ \boldsymbol{q}_P = \boldsymbol{N}\boldsymbol{\lambda}$
	$\boldsymbol{f} = -\boldsymbol{N}^T\boldsymbol{s} + \boldsymbol{k}_0$
	$\boldsymbol{f} \geqslant \boldsymbol{0},\ \boldsymbol{\lambda} \geqslant \boldsymbol{0},\ \boldsymbol{f}^{\mathbf{T}}\boldsymbol{\lambda} = 0$
kinematics	$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{q}$
equilibrium	$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0,\ \boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{r}$
kinematic load	$\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{w}_{\mathbf{0}}$

Reduced system of governing equations:

	$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$	$\boldsymbol{w}_p$	$\boldsymbol{w}_w$	$\boldsymbol{s}$	$\boldsymbol{r}$	1	
$\nabla\boldsymbol{L}_{\boldsymbol{\lambda}} =$				$-\boldsymbol{N}^{-T}$		$\boldsymbol{k}_0$	$\geqslant \boldsymbol{0}$
$\nabla\boldsymbol{L}_{w_p}$				$\boldsymbol{C}_p^T$		$-\boldsymbol{p}_0$	$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_{w_w}$				$\boldsymbol{C}_w^T$	$-\boldsymbol{I}$		$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_s$	$-\boldsymbol{N}$	$\boldsymbol{C}_p$	$\boldsymbol{C}_w$	$-\boldsymbol{E}^{-1}$			$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_r$			$-\boldsymbol{I}$			$\boldsymbol{w}_0$	$= \boldsymbol{0}$

Potential:

$$L(\boldsymbol{\lambda}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r}) = -\frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} - \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{N}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{I}\boldsymbol{r} + \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{k}_0 - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_0 + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0$$

Saddle point:

$$L(\boldsymbol{\lambda}_*, \boldsymbol{w}_{\mathbf{p}*}, \boldsymbol{w}_{\mathbf{w}*}, \boldsymbol{s}_*, \boldsymbol{r}_*) = \min_{\boldsymbol{\lambda}\geqslant\boldsymbol{0}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}} \max_{\boldsymbol{s},\boldsymbol{r}} L(\boldsymbol{\lambda}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r})$$

Primal problem:

find $\min\limits_{\boldsymbol{\lambda}, \boldsymbol{w}_{\mathbf{p}}, \boldsymbol{w}_{\mathbf{w}}, \boldsymbol{s}} \left\{ \frac{1}{2}\boldsymbol{s}^{\mathbf{T}}\boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} + \boldsymbol{\lambda}^T\boldsymbol{k}_0 - \boldsymbol{w}_p^{\mathbf{T}}\boldsymbol{p}_0 \right\}$

subject to

$\boldsymbol{C}_{\mathbf{p}}\boldsymbol{w}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\boldsymbol{w}_{\mathbf{w}} - \boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} - \boldsymbol{N}\boldsymbol{\lambda} = \boldsymbol{0}$

$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$

$\boldsymbol{w}_{\mathbf{w}} = \boldsymbol{w}_0$

Dual problem:

find $\max\limits_{\boldsymbol{s},\boldsymbol{r}} \left\{ -\frac{1}{2}\boldsymbol{s}^T\boldsymbol{E}^{-\mathbf{1}}\boldsymbol{s} + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0 \right\}$

subject to

$\boldsymbol{N}^T\boldsymbol{s} \leqslant \boldsymbol{k}_0$

$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$

$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{r} = \boldsymbol{0}$

Table 4
Rigid-perfectly plastic structure under static and kinematic load

Governing relations:

constitutive	$\dot{\boldsymbol{q}} = \boldsymbol{N}\dot{\boldsymbol{\lambda}}$
	$\boldsymbol{f} = -\boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{k}_0$
	$\boldsymbol{f} \geqslant \boldsymbol{0},\ \dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0},\ \boldsymbol{f}^{\mathbf{T}}\dot{\boldsymbol{\lambda}} = 0$
kinematics	$\boldsymbol{C}_{\mathbf{p}}\dot{\boldsymbol{w}}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\dot{\boldsymbol{w}}_{\mathbf{w}} = \dot{\boldsymbol{q}}$
equilibrium	$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0,\ \boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{r}$
kinematic load	$\dot{\boldsymbol{w}}_{\mathbf{w}} = \dot{\boldsymbol{w}}_{\mathbf{0}}$

Reduced system of governing equations:

	$\boldsymbol{\lambda} \geqslant \boldsymbol{0}$	$\dot{\boldsymbol{w}}_p$	$\dot{\boldsymbol{w}}_w$	$\boldsymbol{s}$	$\boldsymbol{r}$	1	
$\nabla\boldsymbol{L}_{\dot{\boldsymbol{\lambda}}} =$				$-\boldsymbol{N}^T$		$\boldsymbol{k}_0$	$\geqslant \boldsymbol{0}$
$\nabla\boldsymbol{L}_{\dot{w}_p}$				$\boldsymbol{C}_p^T$		$-\boldsymbol{p}_0$	$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_{\dot{w}_w}$				$\boldsymbol{C}_w^T$	$-\boldsymbol{I}$		$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_s$	$-\boldsymbol{N}$	$\boldsymbol{C}_p$	$\boldsymbol{C}_w$				$= \boldsymbol{0}$
$\nabla\boldsymbol{L}_r$			$-\boldsymbol{I}$			$\boldsymbol{w}_0$	$= \boldsymbol{0}$

Potential:

$$L(\dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{w}}_{\mathbf{p}}, \dot{\boldsymbol{w}}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r}) = -\boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} + \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{w}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{I}\boldsymbol{r} + \boldsymbol{\lambda}^{\mathbf{T}}\boldsymbol{k}_0 - \boldsymbol{w}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{p}_0 + \boldsymbol{r}^{\mathbf{T}}\boldsymbol{w}_0$$

Saddle point:

$$L(\dot{\boldsymbol{\lambda}}_*, \dot{\boldsymbol{w}}_{\mathbf{p}*}, \dot{\boldsymbol{w}}_{\mathbf{w}*}, \boldsymbol{s}_*, \boldsymbol{r}_*) = \min_{\dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0}, \dot{\boldsymbol{w}}_{\mathbf{p}}, \dot{\boldsymbol{w}}_{\mathbf{w}}} \max_{\boldsymbol{s},\boldsymbol{r}} L(\dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{w}}_{\mathbf{p}}, \dot{\boldsymbol{w}}_{\mathbf{w}}, \boldsymbol{s}, \boldsymbol{r})$$

Primal problem:

find $\min_{\dot{\boldsymbol{\lambda}}\dot{\boldsymbol{w}}_{\mathbf{p}}, \dot{\boldsymbol{w}}_{\mathbf{w}},} \{\dot{\boldsymbol{\lambda}}^{\mathbf{T}}\boldsymbol{k}_0 - \dot{\boldsymbol{w}}_p^{\mathbf{T}}\boldsymbol{p}_0\}$

subject to

$$\boldsymbol{C}_{\mathbf{p}}\dot{\boldsymbol{w}}_{\mathbf{p}} + \boldsymbol{C}_{\mathbf{w}}\dot{\boldsymbol{w}}_{\mathbf{w}} - \boldsymbol{N}\dot{\boldsymbol{\lambda}} = \boldsymbol{0}$$
$$\dot{\boldsymbol{\lambda}} \geqslant 0$$
$$\dot{\boldsymbol{w}}_{\mathbf{w}} = \dot{\boldsymbol{w}}_0$$

Dual problem:

find $\max_{\boldsymbol{s},\boldsymbol{r}}\{\boldsymbol{r}^T\dot{\boldsymbol{w}}_0\}$

subject to

$$\boldsymbol{N}^T\boldsymbol{s} \leqslant \boldsymbol{k}_0$$
$$\boldsymbol{C}_{\mathbf{p}}^{\mathbf{T}}\boldsymbol{s} = \boldsymbol{p}_0$$
$$\boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{r} = \boldsymbol{0}$$

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find $\max_{\hat{\boldsymbol{r}},\boldsymbol{s}}\{\hat{\boldsymbol{r}}^{\mathbf{T}}\dot{\boldsymbol{w}}_0\}$

subject to

$$\boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} \leqslant \boldsymbol{k}_0 \qquad \boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - \boldsymbol{B}^{\mathbf{T}}\hat{\boldsymbol{r}} = \boldsymbol{0}. \tag{35}$$

Finally, $\boldsymbol{r}$ may be given up to an unknown factor r°:

$$\boldsymbol{r} = r^\circ\boldsymbol{r}_{\mathbf{0}} \tag{36}$$

were $\boldsymbol{r}_{\mathbf{0}}$ is a prescribed reference load. Then model (31)–(32) reduces to the well known theorems concerning the ultimate load factor:

find $\min_{\dot{\boldsymbol{\lambda}},\dot{\boldsymbol{w}}_{\mathbf{w}}}\{\dot{\boldsymbol{\lambda}}^{\mathbf{T}}\boldsymbol{k}_0\}$

subject to

$$\boldsymbol{C}_{\mathbf{w}}\dot{\boldsymbol{w}}_{\mathbf{w}} - \boldsymbol{N}\dot{\boldsymbol{\lambda}} = \boldsymbol{0} \qquad \dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0} \qquad \boldsymbol{r}_{\mathbf{0}}^{\mathbf{T}}\dot{\boldsymbol{w}}_{\mathbf{w}} = 1 \tag{37}$$

find $\max_{\boldsymbol{s},r^\circ} r^\circ$

subject to

$$\boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} \leqslant \boldsymbol{k}_0 \qquad \boldsymbol{C}_{\mathbf{w}}^{\mathbf{T}}\boldsymbol{s} - r^\circ\boldsymbol{r}_{\mathbf{0}} = \boldsymbol{0}. \tag{38}$$

Thinking in terms of optimum limit load, we may replace $\boldsymbol{r}$ in the above models by $\boldsymbol{p}$ and drop the subscript for matrix $\boldsymbol{C}$ and vector $\dot{\boldsymbol{w}}$. Then, model (37)–(38) obtains more conventional form:

$$\text{find } \min_{\dot{\boldsymbol{\lambda}},\dot{\boldsymbol{w}}}\{\dot{\boldsymbol{\lambda}}^{\mathbf{T}}\boldsymbol{k}_0\}$$

subject to

$$\begin{aligned} &\boldsymbol{C}\dot{\boldsymbol{w}} - \boldsymbol{N}\dot{\boldsymbol{\lambda}} = \boldsymbol{0} \\ &\dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0} \\ &\boldsymbol{p}_\mathbf{0}^{\mathbf{T}}\dot{\boldsymbol{w}} = 1 \end{aligned} \tag{39}$$

$$\text{find } \max_{\boldsymbol{s},p^\circ} p^\circ$$

subject to

$$\begin{aligned} &\boldsymbol{N}^{\mathbf{T}}\boldsymbol{s} \leqslant \boldsymbol{k}_0 \\ &\boldsymbol{C}^{\mathbf{T}}\boldsymbol{s} - p^\circ\boldsymbol{p}_\mathbf{0} = \boldsymbol{0}. \end{aligned} \tag{40}$$

8. Conclusion

Finite-dimensional models presented in this part of the paper can be used directly in numerical analysis of structures. The algorithms solving quadratic and linear problems of mathematical programming are efficient enough to compete with iterative methods of solving systems of linear equations and inequalities. Moreover, the models derived by means of the proposed methodology are strict within the frame of adopted assumptions. Using them one avoids implicit and often misleading simplifications that arise inevitably when linear complementarity problems are ad hoc replaced by iterations performed, for example, on the set of Eq. (13) with variable stiffness matrix.