BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 52, No. 2, 2004

Mirror image property for the optimal solutions of two single processor scheduling problems with due intervals determination

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Abstract. In the paper, we investigate two single processor problems, which deal with the process of negotiation between a producer and a customer about delivery time of final products. This process is modelled by a due interval, which is a generalization of well known classical due date and describes a time interval, in which a job should be finished. In this paper we consider two different mathematical models of due intervals. In both considered problems we should find such a schedule of jobs and such a determination of due intervals to each job, that the generalized cost function is minimized. The cost function is the maximum of the following three weighted parts: the maximum tardiness, the maximum earliness and the maximum due interval size. For the first problem we proved several properties of its optimal solution and next we show the mirror image property for both of considered problems, which helps us to provide an optimal solution for the second problem.

Keywords: scheduling, processor, due interval, cost criterion.

1. Introduction

The paper deals with scheduling problems which model the process of negotiation between the producer and the customer about the delivery time of the final products. The producer objective usually is to have the latest time of delivering products, while the customer tries to have them as soon as possible. The compromise of this negotiation is a time period in which the products should be completed by producer and available to be taken by customer. This situation can be modelled in scheduling problems by a due interval, which is an extension of the classical due date and describes a time interval, in which a job should be finished.

The extensive surveys of the results obtained for the due date determination problems can be found in Refs. 1–3. The various models of the due interval determination in scheduling problems with general sum-type criterion have been considered in several papers [4–8]. In the papers [7] and [6] the authors focused on the optimal, common for all the jobs, due interval assignment in some single and parallel processors scheduling problems, respectively. In their model it is assumed that the size of the due interval is *a priori* given. The model of due interval considered in [6] and [7] have been extended in [4], [5] and [8]. This extension concerns the size of due interval, which is as well as its location a decision variable.

To be more precise, in this paper we consider two scheduling problems. In the first one we consider the due interval, called "constant due interval", which is common for all the jobs. In the second one we consider due intervals, called "slack due intervals", which are common for all the jobs with the identical processing times. For the considered problems, we should find a schedule of jobs, due intervals and their locations such that their criterion values are minimized. We minimize the cost type criterion which is the maximum of the following three weighted parts: the maximum tardiness, the maximum earliness and the maximum due interval size. We prove some properties of the optimal solutions to both problems.

The remaining part of the paper is organized as follows. In the next section, we give a precise formulation of the considered problems. The optimal solution properties of the first problem are presented in Section 3. Section 4 deals with the mirror image property of the considered problems and some properties of the optimal solution to the second problem. Some final remarks are given in Section 5.

2. Formulation of the problems

We consider two scheduling problems (P_k and P_q), in which there is given a set $J = \{1, \ldots, n\}$ of n independent and non-preemptive jobs to be scheduled on a single processor. At the moment, the processor can process only one job. We assume that processor executes the jobs without idle times. For each job its processing time p_j is given and we will consider the following two models of the job due intervals $\langle d'_j; d''_j\rangle$ (for problems P_k and P_q, respectively):

- constant due interval — $\langle d'_j = k_1; d''_j = k_2\rangle$, and
- slack due interval — $\langle d'_j = p_j + q_1; d''_j = p_j + q_2\rangle$,

where k_1, k_2 ($k_1 \leqslant k_2$) and q_1, q_2 ($q_1 \leqslant q_2$) denote common due interval parameters for P_k and P_q, respectively.

The problems P_k and P_q consist in finding such schedules π ($\pi = \{S_j : j \in J\}$, where S_j denotes the starting moment of job j) and such values of the parameters k_1, k_2 and q_1, q_2, respectively, for P_k and P_q, which minimize the following criterion:

$$f(\pi, d'_j, d''_j) = \max(A \max_{j\in J} E_j, \max_{j\in J} B(d''_j - d'_j), C \max_{j\in J} T_j),$$

where: $E_j = \max(d'_j - C_j, 0)$, $T_j = \max(0, C_j - d''_j)$, $\max(d''_j - d'_j)$, C_j is, respectively, the earliness, the tardiness, the due interval size, the completion moment of the

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job j, and A, B, C are some positive cost weights.

Finally, the criterion reduces to the following ones, for problem P_k and P_q, respectively:

$$f_k(\pi, k_1, k_2) = \max(A \max_{j\in J} E_j, B(k_2 - k_1), C \max_{j\in J} T_j) \quad (1)$$

and

$$f_q(\pi, q_1, q_2) = \max(A \max_{j\in J} E_j, B(q_2 - q_1), C \max_{j\in J} T_j). \quad (2)$$

3. Optimal solution properties of problem P_k

In the sequel for a given π we assume that: $C_{\min}(\pi) = \min_{j\in J} C_j(\pi)$ and $C_{\max}(\pi) = \max_{j\in J} C_j(\pi)$.

PROPERTY 1. For a given schedule π of P_k, the optimal values of the parameters $k_1^*(\pi)$ and $k_2^*(\pi)$ fulfill, respectively, the following inequalities: $k_1^*(\pi) \geqslant C_{\min}(\pi)$ and $k_2^*(\pi) \leqslant C_{\max}(\pi)$.

Proof. Assume that π is a schedule in P_k, where the inequalities $k_1^*(\pi) \geqslant C_{\min}(\pi)$ and $k_2^*(\pi) \leqslant C_{\max}(\pi)$ are not satisfied. There are two cases, which should be considered for π, namely for a given value of k_2 or k_1 ($k_2 \geqslant k_1$), the optimal value of $k_1^*(\pi)$ or $k_2^*(\pi)$ is equal to
1° $k_1'(\pi) = C_{\min}(\pi) - \varepsilon$ or
2° $k_2'(\pi) = C_{\max}(\pi) + \varepsilon$, respectively, where ε is some positive value, ($\varepsilon > 0$).

Ad 1°. Let $f_k(\pi, k_1'(\pi), k_2)$ and $f_k(\pi, k_1''(\pi), k_2)$ denote the values of the criterion (1) obtained for the values of the parameters: $k_1'(\pi) = C_{\min}(\pi) - \varepsilon$ and $k_1''(\pi) = C_{\min}(\pi)$, respectively. We have (a).
Analogical result we can obtain for the case 2°.

These results contradict the assumptions that k_1' and k_2' are optimal. Thus, it follows from the above considerations, that the optimal values of the parameters $k_1^*(\pi)$ and $k_2^*(\pi)$ should satisfy the inequalities: $k_1^*(\pi) \geqslant C_{\min}(\pi)$ and $k_2^*(\pi) \leqslant C_{\max}(\pi)$. □

It follows from Lemma 1, that the criterion value (1) is equal to Eq. (3), shown at the bottom of the page.

Now we prove some lemma for the following general function $h : \Re^3 \to \Re$ (see that (3) is a special case of this function):

$$h(u, v, w) = \max(A_1 u, A_2 v, A_3 w) \quad (4)$$

subject to:

$$u + v + w = A, \quad (5)$$

where u, w and v are some nonnegative variables, A is a given nonnegative constant and A_1, A_2 and A_3 are given nonnegative weights.

LEMMA 1. If $A_1 u = A_2 v = A_3 w$, then the value of the expression (4) is minimal and the following values of the variables u, w, v minimize (4):

$$u^* = \frac{A_2 A_3 A}{A_1 A_2 + A_1 A_3 + A_2 A_3},$$
$$v^* = \frac{A_1 A_3 A}{A_1 A_2 + A_1 A_3 + A_2 A_3},$$
$$w^* = \frac{A_1 A_2 A}{A_1 A_2 + A_1 A_3 + A_2 A_3}. \quad (6)$$

Proof. Let h' denote the value of the function (4) obtained for the following values of the variables u', v', w', for which $A_1 u' = A_2 v' = A_3 w'$.

Assume now that the value of at least one variable u, w or v, let's say u, is smaller than value u'. It follows from the constraint (5) that in this case the value of at least one from the remaining variable, let's say v, has to be greater than value v'. It means that the expression (4) can be estimated by: $h(u, v, w) = \max(A_1 u, A_2 v, A_3 w) > h' = \max(A_1 u', A_2 v', A_3 w')$, which ends the first part of the proof.

Basing on the above considerations and the constraint (5), we can easily calculate u^*, v^*, w^* (i.e. (6)) solving the following system of equations:

$$\begin{cases} A_1 u^* = A_2 v^* \\ A_2 v^* = A_3 w^* \\ u^* + v^* + w^* = A. \end{cases}$$

□

Let $C_{\pi(j)}$ and $p_{\pi(j)}$ denote, respectively, the completion moment and the processing time of the job placed on the position j in the schedule π.

Based on Lemma 1 and the expression (3), for the schedule π and the optimal values of $k_1^*(\pi)$ and $k_2^*(\pi)$ the criterion (1) can be reformulated as follows:

$$\begin{aligned} f_k(\pi, k_1'(\pi), k_2) &= \max(A \max_{j\in J} E_j, B(k_2 - k_1'(\pi)), C \max_{j\in J} T_j) = \max(A \max_{j\in J}(\max(d_j' - C_j(\pi), 0)), B(k_2 - k_1'(\pi)), C \max_{j\in J} T_j) \\ &= \max(A \max(k_1'(\pi) - C_{\min}(\pi), 0), B(k_2 - k_1'(\pi)), C \max_{j\in J} T_j) \\ &= \max(0, B(k_2 - C_{\min}(\pi) + \varepsilon), C \max_{j\in J} T_j) \\ &\geqslant \max(0, B(k_2 - C_{\min}(\pi)), C \max_{j\in J} T_j) = f_k(\pi, k_1''(\pi), k_2). \end{aligned} \quad \text{(a)}$$

$$\begin{aligned} f_k(\pi, k_1^*(\pi), k_2^*(\pi)) &= \max(A \max_{j\in J}(\max(k_1^*(\pi) - C_j(\pi), 0)), B(k_2^*(\pi) - k_1^*(\pi)), C \max_{j\in J}(\max(0, C_j(\pi) - k_2^*(\pi)))) \\ &= \max(A(k_1^*(\pi) - C_{\min}(\pi)), B(k_2^*(\pi) - k_1^*(\pi)), C(C_{\max}(\pi) - k_2^*(\pi))). \end{aligned} \quad (3)$$

$$f_k(\pi, k_1^*(\pi), k_2^*(\pi)) = \frac{ABC}{AB+AC+BC}\left(\sum_{j=1}^{n} p_{\pi(j)} - p_{\pi(1)}\right), \tag{7}$$

since $C_{\max}(\pi) = C_{\pi(n)} = \sum_{j=1}^{n} p_{\pi(j)}$ and $C_{\min}(\pi) = C_{\pi(1)} = p_{\pi(1)}$.

Some optimal solution properties for P_k, which concern the optimal schedule of jobs and the optimal values of k_1^* and k_2^*, are given below.

PROPERTY 2. For a given schedule π of jobs, the optimal values of the parameters $k_1^*(\pi)$ and $k_2^*(\pi)$ are equal to:

$$k_1^*(\pi) = \frac{BC\sum_{j=1}^{n} p_j + A(B+C)p_{\pi(1)}}{AB+AC+BC} \quad \text{and}$$
$$k_2^*(\pi) = \frac{C(A+B)\sum_{j=1}^{n} p_j + ABp_{\pi(1)}}{AB+AC+BC}. \tag{8a}$$

P r o o f . Basing on Lemma 1 and the expression (3), we have Eq. (8b), shown at the bottom of the page.

Since $C_{\min}(\pi) = C_{\pi(1)} = p_{\pi(1)}$ and $C_{\max}(\pi) = C_{\pi(n)} = \sum_{j=1}^{n} p_{\pi(j)}$, thus we obtain the first formula of (8a).

In similar way we can obtain the second formula of (8a). □

PROPERTY 3. There exists an optimal solution (i.e. (π^*, k_1^*, k_2^*)) to P_k, in which the job with the largest processing time is performed as the first one.

P r o o f . It follows form (7) that the criterion value (1) is minimal, if the job with the largest processing time is executed on the first position. □

The optimal algorithm solving P_k can be realized in $O(n)$ time, since the values of the parameters k_1^* and k_2^* depend only on the value of the processing time of the job which is executed on the first position of π^*, i.e., the job with the largest processing time and the remaining jobs can be scheduled in π^* in an arbitrary order.

4. Mirror image of the optimal solutions to problems P_k and P_q

In the sequel, we will use the following notation. An upper index k and q will indicate the values of the problem P_k and P_q parameters, respectively. Now we consider the following mirror image property.

THEOREM 1. If $A^q = C^k$, $B^q = B^k$, $C^q = A^k$, then the optimal schedule of the problem P_k or P_q can be obtained from an optimal schedule of the other problem (P_q or P_k) by reversing the order of the jobs on the processor, and determining the appropriate due interval parameters from the following equations: $\sum_{j=1}^{n} p_j = q_1 + k_2 = q_2 + k_1$. Moreover, the optimal criterion values for both problems are equal.

P r o o f . Assume π^{rev} denotes a schedule, in which the jobs are executed in the reversed order on the processor with respect to the schedule π. It is easy to see that the makespan value ($C_{\max}$) for both schedules is the same and equal to $C_{\max} = \sum_{j=1}^{n} p_j$. From the reversing execution of the jobs in the schedule π it follows that for π^{rev} we have:

$$S_j(\pi^{rev}) = C_{\max} - S_j(\pi) - p_j \quad \text{for } j = 1, \ldots, n \tag{9}$$

(see also an example given in Fig. 1).

To prove the theorem, at first we need to show that for any π and reversing to it π^{rev} the following equality holds:

$$\begin{aligned} &f_k(\pi, k_1, k_2) \\ &= \max\left(A^k \max_{j\in J} E_j^k(\pi), B^k(k_2 - k_1), C^k \max_{j\in J} T_j^k(\pi)\right) \\ &= \max\left(A^q \max_{j\in J} E_j^q(\pi^{rev}), B^q(q_2 - q_1)_j, C^q \max_{j\in J} T_j^q(\pi^{rev})\right) \\ &= f_q(\pi^{rev}, q_1, q_2). \end{aligned} \tag{10}$$

According to the expressions (9) and $C_{\max} = \sum_{j=1}^{n} p_j = q_1 + k_2 = q_2 + k_1$, we can formulate the following equations:

$$\begin{aligned} E_j^q(\pi^{rev}) &= \max\left(d'^q_j - C_j(\pi^{rev}), 0\right) \\ &= \max\left(p_j + q_1 - S_j(\pi^{rev}) - p_j, 0\right) \\ &= \max\left(q_1 - S_j(\pi^{rev}), 0\right) = \\ &= \max\left(C_{\max} - k_2 - C_{\max} + S_j(\pi) + p_j, 0\right) \\ &= \max\left(C_j(\pi) - k_2, 0\right) \\ &= \max\left(C_j(\pi) - d''^k_j, 0\right) = T_j^k(\pi), \end{aligned}$$

$$\begin{aligned} T_j^q(\pi^{rev}) &= \max\left(0, C_j(\pi^{rev}) - d''^q_j\right) \\ &= \max\left(0, S_j(\pi^{rev}) + p_j - p_j - q_2\right) \\ &= \max\left(0, S_j(\pi^{rev}) - q_2\right) \\ &= \max\left(0, C_{\max} - S_j(\pi) - p_j - C_{\max} + k_1\right) \\ &= \max\left(0, k_1 - C_j(\pi)\right) \\ &= \max\left(0, d'^k_j - C_j(\pi)\right) = E_j^k(\pi), \end{aligned}$$

$$q_2 - q_1 = C_{\max} - k_1 - (C_{\max} - k_2) = k_2 - k_1.$$

$$\left(u^* = \frac{A_2A_3A}{A_1A_2 + A_1A_3 + A_2A_3}\right) \Rightarrow \left(k_1^*(\pi) - C_{\min}(\pi) = \frac{BC}{AB+AC+BC}\left(C_{\max}(\pi) - C_{\min}(\pi)\right)\right). \tag{8b}$$

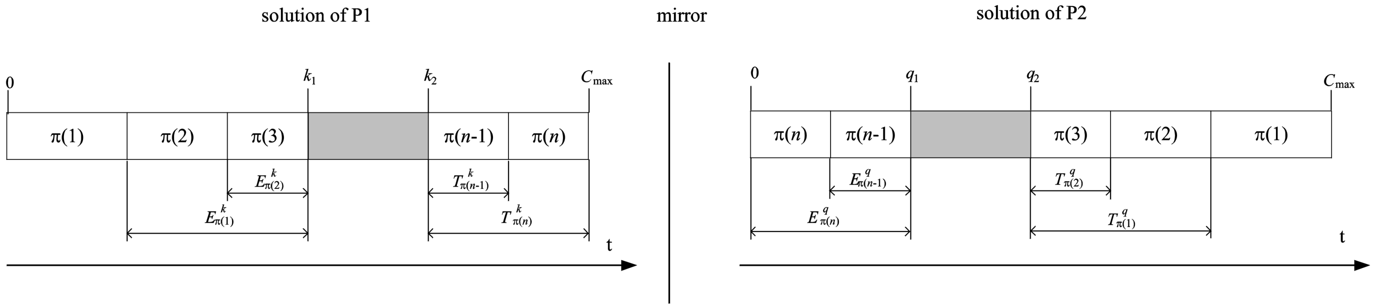

Fig. 1. Mirror image of solutions to P_k and P_q

Since $A^q = C^k$, $B^q = B^k$, $C^q = A^k$, then the above equations imply that the equation (10) is satisfied (see also Fig. 1).

Let us pass to proving, that if (π^*, k_1^*, k_2^*) is the optimal solution to P_k, than the corresponding solution (π^{*rev}, q_1^*, q_2^*) to P_q is also optimal, and vice versa.

Assume that (π^*, k_1^*, k_2^*) is the optimal solution to P_k and the equation (10) is satisfied and the solution (π^{*rev}, q_1^*, q_2^*) is not the optimal one to P_q. Then, there exists a solution (π', q_1', q_2') such that $f_q(\pi', q_1', q_2') < f_q(\pi^{*rev}, q_1^*, q_1^*)$. Observe that ($\pi'$, q_1', q_2') is a solution reversed with respect to some solution (π'^{rev}, k_1', k_2') to P_k. Then, according to (10), we must have $f_k(\pi'^{rev}, k_1', k_2') = f_q(\pi', q_1', q_2') < f_q(\pi^{*rev}, q_1^*, q_2^*) = f_k(\pi^*, k_1^*, k_2^*)$, which contradicts the optimality of (π^*, k_1^*, k_2^*).

Similar result can be obtained if we assume that (π^{*rev}, q_1^*, q_2^*) is the optimal solution to P_q and the solution (π^*, k_1^*, k_2^*) is not the optimal one to P_k. □

Below, we present the properties of the optimal solution of P_q, which concern the optimal schedule of jobs and the optimal values of q_1 and q_2.

PROPERTY 4. For any schedule π of jobs in P_q, the optimal values of the parameters q_1^* and q_2^*, are equal to

$$q_1^*(\pi) = \frac{BC\left(\sum_{j=1}^{n} p_j - p_{\pi(n)}\right)}{AB + AC + BC} \quad \text{and}$$

$$q_2^*(\pi) = \frac{C\,(A+B)\left(\sum_{j=1}^{n} p_j - p_{\pi(n)}\right)}{AB + AC + BC},$$

and there exists an optimal solution to P_q, where the job with the largest processing time is performed as the last one.

P r o o f . It follows immediately from Theorem 1 and Property 2. □

The optimal algorithm solving P_q can be realized in $O(n)$ time, since the values of the parameters q_1^* and q_2^* depend on the processing time of the job which is executed on the last position, i.e., the job with the largest processing time and the remaining jobs can be scheduled in π^* in an arbitrary order.

5. Final remarks

We considered two problems of scheduling jobs on a single processor where a due interval should be assigned to each job such that the maximum of the following criterion weighted parts: the maximum tardiness, the maximum earliness and the due interval size is minimized. We established some properties of an optimal solution to the considered problems. The most important result is Theorem 1, which concerns the mirror image of the solutions of the considered problems. It can be adopted for an instance to the sum type criterion function. Moreover, Theorem 1 can be also extended to the problems of scheduling jobs on parallel processors.

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