BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 52, No. 2, 2004

Stabilisation of LC ladder network

W. MITKOWSKI*

Institute of Automatics, University of Mining and Metallurgy, 30 Mickiewicza Ave., 30-059 Kraków, Poland

Abstract. In this paper stabilisation problem of LC ladder network is established. We studied the following cases: stabilisation by inner resistance, by velocity feedback and stabilisation by dynamic linear feedback, in particularly stabilisation by first range dynamic feedback. The global asymptotic stability of the respectively system is proved by LaSalle's theorem. In the proof the observability of the dynamic system plays an essential role. Numerical calculations were made using the Matlab/Simulink program.

Keywords: ladder network, feedback stabilisation, asymptotic stability.

1. Introduction

We consider an electric ladder network of the L and GC-type shown in Fig. 1. The parameters of the network $L > 0$, $G_i > 0$ and $C > 0$ are known, where $i = 1, 2$.

The system shown in Fig. 1 is described by following equations:

$$\ddot{x}(t) + D\dot{x}(t) + Ax(t) = Bu(t),\quad x(t) = [x_1(t)\ x_2(t)\ \ldots\ x_n(t)]^T \tag{1}$$

where A is tridiagonal matrix, D is diagonal matrix,

$$A = \omega^2 \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} g_1 & 0 & 0 & \ldots & 0 \\ 0 & g_2 & 0 & \ldots & 0 \\ 0 & 0 & g_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & g_2 \end{bmatrix},$$

$$B = \omega^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \omega^2 = \frac{1}{LC},$$

$$g_1 = \frac{G_1}{C},\ g_2 = \frac{G_2}{C}. \tag{2}$$

From (1) we have

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -D \end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t). \tag{3}$$

Remark 1. The eigenvalues of matrix A given in (2) are (see for example [1]) given by the following equation:

$$\lambda_i(A) = 2\omega^2(1 - \cos\varphi_i) = 4\omega^2 \sin^2\frac{\varphi_i}{2},\quad \varphi_i = \frac{i\pi}{n+1}, i = 1, 2, ..., n \tag{4}$$

From (4) we have $\lambda_i(A) > 0$. Thus $\det A \neq 0$ and $A = A^T$ is positive definite matrix.

Let

$$P = \sqrt{\frac{2}{n+1}}\begin{bmatrix} \sin\varphi_1 & \sin 2\varphi_1 & \ldots & \sin n\varphi_1 \\ \sin\varphi_2 & \sin 2\varphi_2 & \ldots & \sin n\varphi_2 \\ \ldots & \ldots & \ldots & \ldots \\ \sin\varphi_n & \sin 2\varphi_n & \ldots & \sin n\varphi_n \end{bmatrix}, \tag{5}$$

where φ_i is given in (4). From (5) we obtain $P^2 = I$. Thus $P^{-1} = P$ and $PAP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i is given in (4).

2. LC electric ladder network

If $G_1 = G_2 = 0$ [1,2], then we have ladder network of LC type (see Fig. 1). In this case our network is undamped second order system described by (1) or (3) with $D = 0$.

The system (1) with $D = 0$ is diagonalizable [1]:

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = f_i(t),\ f_i(t) = \omega^2\sqrt{\frac{2}{n+1}}u(t)\sin\varphi_i, \tag{6}$$

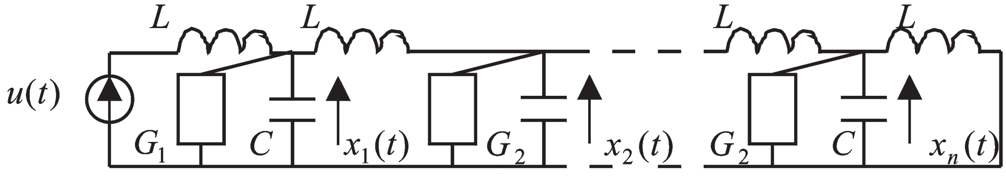

Fig. 1. Electric ladder network of the L and GC-type

* e-mail: wmi@ia.agh.edu.pl

where $\omega_i = \sqrt{\lambda_i(A)} = 2\omega \sin \frac{\varphi_i}{2}$, $i = 1, 2, ..., n$, φ_i is given in (4). Solution of the equation (6) with $D = 0$ is given by (see for example [3,4]) following formula:

$$z_i(t) = \cos(\omega_i t)\, z_i(0) + \omega_i^{-1} \sin(\omega_i t)\, \dot{z}_i(0) + \omega_i^{-1} \int_0^t \sin(\omega_i(t-\tau)) f_i(\tau) d\tau. \quad (7)$$

Using (7) the solution of the equation (1) with $D = 0$ can be represented in the following form:

$$x(t) = Pz(t), \;\; z(t) = [z_1(t) \dots z_n(t)]^T, \quad (8)$$

where P is given in (5).

Remark 2. The system (1) with $D = 0$ is controllable (see for example [5]) if and only if the pair $(A; B)$ is controllable. The matrix A given in (2) is diagonalizable (see *Remark* 1). Thus the pair $(P^{-1}AP; P^{-1}B)$ is controllable and consequently the pair $(A; B)$ is controllable.

Remark 3. Let Q be a real matrix $m \times n$. It is obvious, that the pair $(Q; A)$ is observable if and only if the rang $Z = n$, $Z = [\,Q^T \quad A^T Q^T \quad \dots \quad (A^T)^{n-1} Q^T\,]$. Similarly it is obvious, that the pair $(Q; A)$ is observable if and only if the rang $M(s) = n$ for any complex number s, where

$$M(s) = \begin{bmatrix} sI - A \\ Q \end{bmatrix}. \quad (9)$$

Consequently the pair $(Q; A)$ is observable if and only if the equation $M(s)\nu = 0$ has no nonzero solution ν for any complex number s (criterion of Hautus 1969, see for example [1]).

Remark 4. Let $y(t) \in R^m$ be the output of the system (1) with $D = 0$. Let $y(t) = Qx(t)$ or $y(t) = Q\dot{x}(t)$. Let

$$M_1 = \begin{bmatrix} Q & 0 \\ 0 & Q \\ QA & 0 \\ 0 & QA \\ QA^2 & 0 \\ 0 & QA^2 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ 0 & QA^{n-1} \end{bmatrix}, \; M_2 = \begin{bmatrix} 0 & Q \\ QA & 0 \\ 0 & QA \\ QA^2 & 0 \\ 0 & QA^2 \\ QA^3 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ QA^n & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & Q \\ Q & 0 \\ 0 & QA \\ QA & 0 \\ 0 & QA^2 \\ QA^2 & 0 \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ QA^{n-1} & 0 \end{bmatrix}, \; M_4 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \quad (10)$$

where Q is a real matrix $m \times n$. The pair $\left([Q \;\, 0]; \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}\right)$ is observable if and only if rang $M_1 = 2n$. Similarly, the pair $\left([0 \quad Q]; \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}\right)$ is observable if and only if rang $M_2 = 2n$. Let $(Q; A)$ be observable. Thus from *Remark* 3 rang $Z = n$, $Z = [Q^T \;\, A^T \;\, Q^T \;\, \dots \;\, (A^T)^{n-1} \;\, Q^T]$ and consequently rang $M_1 = 2n$. Next we are going to show, that rang $M_2 = 2n$. From (10) $M_2 = M_3 M_4$, where $\det M_4 \neq 0$, because $\det A \neq 0$ (see *Remark* 1). Thus rang M_2 = rang M_3. But rang $M_3 = 2n$, because rang $Z = n$. Thus if $(Q; A)$ is observable, we obtain rang $M_2 = 2n$. Consequently, if $(Q; A)$ is observable, then the system $\ddot{x} + Ax = 0$, $y = Qx$ is observable and the system $\ddot{x} + Ax = 0$, $y = Q\dot{x}$ is also observable. □

Remark 5. The eigenvalues of the state matrix of the system (3) with $D = 0$ are given by following formulas: $\pm j\omega_i$, $j^2 = -1$, $\omega_i = \sqrt{\lambda_i(A)}$, $i = 1, 2, ..., n$. For $n = 2$ and $LC = 1$ we have $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$. For $n = 5$ and $LC = 1$ we have $\omega_1 = \sqrt{2 - \sqrt{3}}$, $\omega_2 = 1$, $\omega_3 = \sqrt{2}$, $\omega_4 = \sqrt{3}$, $\omega_5 = \sqrt{2 + \sqrt{3}}$. Thus in the system (1) with $D = 0$ could appear almost periodic oscillations. □

The system (1) with $D = 0$ is stable (see (6)), but not asymptotically stable. Is evident, that the system (1) with $D = 0$ and with static feedback $u(t) = -Ky(t)$ is not asymptotically stable. Thus our question is: how to stabilise the system (1) with $D = 0$?

3. Stabilisation by inner resistance

Now we prove, that the system (1) with $D = 0$ can be stabilised by inner resistance (conductance $G > 0$). We consider two cases: $G_1 = G_2 = G > 0$ and $G_1 = G$, $G_2 = 0$ (see Fig. 1).

Case 1. Let $G_1 = G_2 = G > 0$. Using the transformation (8) from (1) we obtain

$$\ddot{z}_i(t) + g\dot{z}_i(t) + \lambda_i z_i(t) = b_i u(t), \qquad b_i = \omega^2 \sqrt{\frac{2}{n+1}} \sin\varphi_i, g = G/C, \quad (11)$$

where λ_i and φ_i are given by (4) and $\omega^2 = 1/(LC)$. Consequently from (11) we have (see also (3))

$$\frac{d}{dt}\begin{bmatrix} z_i(t) \\ \dot{z}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda_i & -g \end{bmatrix} \begin{bmatrix} z_i(t) \\ \dot{z}_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} u(t). \quad (12)$$

Remark 6. We notice that $\lambda_i > 0$ and $g > 0$. Thus system (11) is asymptotically stable and consequently system (1) with $G_1 = G_2 = G > 0$ is asymptotically stable. □

Remark 7. Let $\Delta = g^2 - 4\lambda \neq 0$, $s_1 = (-g + \sqrt{\Delta})/2$, $s_2 = (-g - \sqrt{\Delta})/2$. Let (see (12) without index "i")

$$Z = \begin{bmatrix} 0 & 1 \\ -\lambda & -g \end{bmatrix}, \quad e^{Zt} = \begin{bmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{bmatrix}. \quad (13)$$

Using simple calculations we obtain

$$\begin{aligned} e_{11}(t) &= \frac{1}{2}[(e^{s_1t}+e^{s_2t})\sqrt{\Delta}-(e^{s_1t}-e^{s_2t})g]/\sqrt{\Delta} \\ e_{12}(t) &= -(e^{s_1t}-e^{s_2t})/\sqrt{\Delta} \\ e_{21}(t) &= \lambda(e^{s_1t}-e^{s_2t})/\sqrt{\Delta} \\ e_{22}(t) &= \frac{1}{2}[(e^{s_1t}+e^{s_2t})\sqrt{\Delta}+(e^{s_1t}-e^{s_2t})g]/\sqrt{\Delta}. \end{aligned} \tag{14}$$

If $\Delta = g^2 - 4\lambda = 0$, thus $g = 2\sqrt{\lambda}$ and $s_1 = s_2 = -g/2 = -\sqrt{\lambda}$. In this case

$$\begin{aligned} e_{11}(t) &= (1+t\sqrt{\lambda})e^{-t\sqrt{\lambda}} \\ e_{12}(t) &= te^{-t\sqrt{\lambda}} \\ e_{21}(t) &= -t\lambda e^{-t\sqrt{\lambda}} \\ e_{22}(t) &= (1-t\sqrt{\lambda})e^{-t\sqrt{\lambda}}. \end{aligned} \tag{15}$$

The solution of the equation (1) with $D \neq 0$ can be obtained from (8). □

Case 2. Now we consider second case. Let $G_1 = G, G_2 = 0$ (see Fig. 1). From (2) we obtain $D = BB^Tg/\omega^4$, where $g = G/C$ and $\omega^2 = 1/(LC)$. The diagonal matrix D is positive semi-definite. The global asymptotic stability of the system (3) with $G_1 = G, G_2 = 0$ is proved by LaSalle's theorem [6]. Let us the Liapunov function (similarly to [6])

$$\begin{aligned} V(x(t),\dot{x}(t)) &= \frac{1}{2}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix}^T \begin{bmatrix} A & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} \\ &= \frac{1}{2}x(t)^TA^Tx(t) + \frac{1}{2}\dot{x}(t)^T\dot{x}(t). \end{aligned} \tag{16}$$

We can notice that $V(x,\dot{x}) > 0$ and $V(x,\dot{x}) \to \infty$ if $[x^T\dot{x}^T] \to \infty$. From (16) and (1) with $u = 0$ we have

$$\frac{d}{dt}V(x(t),\dot{x}(t)) = \dot{V}(x(t),\dot{x}(t)) = -\dot{x}(t)^TD\dot{x}(t). \tag{17}$$

By LaSalle's theorem [6] the solutions of (3) with $u = 0$ asymptotically tends to the maximal invariant subset S of E, where

$$E = \{(x,\dot{x}) : \dot{V} = 0\}. \tag{18}$$

If E contains only the maximal invariant subset $S = \{0\}$, then $x(t) \to 0$ and $\dot{x}(t) \to 0$ if $t \to \infty$. Now we prove that the maximal invariant subset $S = \{0\}$. In the proof the observability of the system (3) with proper output $y(t)$ will play an essential role (see [7]).

The diagonal matrix $D = BB^Tg/\omega^4$ is positive semi-definite and consequently $\dot{V} \leqslant 0$. From $\dot{V} = 0$ we have (see (17) and (18)) $B^T\dot{x}(t) = 0$, because $D = BB^Tg/\omega^4$. We consider following system:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & 0\end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix},\ y(t) = B^T\dot{x}(t). \tag{19}$$

The pair $(B^TP; P^{-1}AP)$ is observable (see (2) and (4), (5)). Thus the pair $(B^T; A)$ is observable and system (19) is observable (see *Remark* 4). The system (19) is observable if and only if (see *Remark* 3) for any complex s the following implication holds:

$$\begin{bmatrix} sI & -I \\ A & sI \\ 0 & B^T\end{bmatrix}\begin{bmatrix} v_1 \\ v_2\end{bmatrix} = 0 \Leftrightarrow \begin{cases} sv_1 - v_2 = 0 \\ Av_1 + sv_2 = 0 \\ B^Tv_2 = 0\end{cases}$$
$$\Rightarrow v_1 = 0 \text{ and } v_2 = 0. \tag{20}$$

Now we consider system (3) with output $y(t) = B^T\dot{x}(t)$ and $u(t) = 0$, i.e. the following system:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -D\end{bmatrix}\begin{bmatrix} x(t) \\ \dot{x}(t)\end{bmatrix}, y(t) = B^T\dot{x}(t). \tag{21}$$

The system (21) is observable if and only if (see *Remark* 3) for any complex s the following implication holds:

$$\begin{bmatrix} sI & -I \\ A & sI+D \\ 0 & B^T\end{bmatrix}\begin{bmatrix} v_1 \\ v_2\end{bmatrix} = 0 \Leftrightarrow \begin{cases} sv_1 - v_2 = 0 \\ Av_1 + sv_2 + Dv_2 = 0 \\ B^Tv_2 = 0\end{cases}$$
$$\Rightarrow v_1 = 0 \text{ and } v_2 = 0. \tag{22}$$

We notice, that if $B^Tv_2 = 0$ then $Dv_2 = 0$, since $D = BB^Tg/\omega^4$. Thus for any complex s the implication (22) is equivalent the implication (20). Therefore for any complex s the implication (22) holds and we obtained following lemma.

LEMMA 1. Let $D = BB^Tg/\omega^4$. Then the system (21) is observable. □

Now we turn to the proof that the maximal invariant subset $S = \{0\} \subseteq E$. From $\dot{V} = 0$ (see (17) and (18)) it results in $B^T\dot{x}(t) = 0$ for $0 \leqslant t$. From Lemma 1, we know that the system (21) is observable. Thus we have $x(t) = 0$ and $\dot{x}(t) = 0$ for $0 \leqslant t$. Consequently we get $E = S = \{0\}$. Summarising we obtained the following theorem.

THEOREM 1. Let $G_1 = G, G_2 = 0$ (see Fig. 1). Then the system (1), (2) or (3), (2) with $u = 0$ is globally asymptotically stable, i.e. the equilibrium point $\{0\}$ is asymptotically stable and its domain of attraction covers the whole space $R^n \times R^n$. □

4. Stabilisation by velocity feedback

Now we consider the undamped second order system (see (1) or (3) with $D = 0$) given in the following form:

$$\ddot{x}(t) + Ax(t) = Bu(t), \quad y(t) = B^Tx(t), \tag{23}$$

where $u(t)$ is the scalar input and $y(t)$ is the scalar output of the system. If

$$u(t) = -K\dot{y}(t), \quad K > 0, \tag{24}$$

then the closed-loop systems becomes

$$\ddot{x}(t) + D\dot{x}(t) + Ax(t) = 0, \quad y(t) = B^T\dot{x}(t), \tag{25}$$

where $D = BB^TK$. For $K = g/\omega^4$ we obtained system (3), (2) with $G_1 = G, G_2 = 0$ and $u = 0$ (see Theorem 1).

Remark 8. The velocity feedback (24) asymptotically stabilises the system (23), (2), i.e. the system (25), (2) with $D = BB^TK$ is globally asymptotically stable. □

5. Stabilisation by dynamic feedback

We consider the system (23) in the following form:

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t), \quad y(t) = \tilde{C}z(t),$$
$$\tilde{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \tilde{C} = [\, B^T \quad 0\,], \tag{26}$$

where $z(t)^T = [\, x(t) \quad \dot{x}(t)\,]$.

The pair $(A; B)$ is controllable (see *Remark* 2). Consequently the pair $(B^T; A)$ is observable, because $A^T = A$. Thus the system (26) is controllable and observable (see *Remark* 4) and a full range Luenberger observer with linear regulator, i.e.

$$\dot{w}(t) = [\tilde{A} - \tilde{G}\tilde{C}]w(t) + \tilde{G}y(t) + \tilde{B}u(t),$$
$$u(t) = \tilde{K}w(t) \tag{27}$$

may be used to stabilise the pair $(z(\cdot), w(\cdot))$. The closed-loop system (26), (27) is given by

$$\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B}\tilde{K} \\ \tilde{G}\tilde{C} & \tilde{A} - \tilde{G}\tilde{C} + \tilde{B}\tilde{K} \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix},$$
$$y(t) = \tilde{C}z(t). \tag{28}$$

From classical result it can be shown that there exist $\tilde{G}$ and $\tilde{K}$ such that the system (28) is globally asymptotically stable (see for example [1]).

The range of the system (27) is equal to $2n$. Similarly we can construct a reduce range Luenberger observer (with range equal $2n-1$).

6. First range dynamic feedback

Now we consider system (26) or (23) with the dynamic feedback given in the following form:

$$u(t) = -K(y(t) + w(t)), \ K > 0,$$
$$\dot{w}(t) = -aw(t) + bu(t), \ a > 0, \ b > 0, \tag{29}$$

where $\dim w(t) = 1$. From (26) and (29) we obtain

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -A & 0 & 0 \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ b \end{bmatrix} u(t),$$
$$s(t) = [\, \alpha B^T \quad 0 \quad \beta\,] \begin{bmatrix} x(t) \\ \dot{x}(t) \\ w(t) \end{bmatrix}, \tag{30}$$

$$u(t) = -Ks(t), \quad K > 0 \tag{31}$$

where $\alpha = 1$, $\beta = 1$ and consequently the closed-loop system can obtain the following form:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -[A + BKB^T] & 0 & -BK \\ -bKB^T & 0 & -[a + bK] \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ w(t) \end{bmatrix}. \tag{32}$$

Remark 9. We notice that for $\alpha \neq 0$ and $\beta \neq 0$ the block-diagonal system (30) with output $s(t)$ is observable. □

THEOREM 2. If $K > 0$, $a > 0$ and $b > 0$, then the closed-loop system (32) is globally asymptotically stable, i.e. $\mathrm{Re}\lambda(\tilde{A}) < 0$, where $\tilde{A}$ is the state matrix of system (32).

Proof. The global asymptotic stability of the closed-loop system (32) is proved by LaSalle's theorem [6]. Consider the Liapunov function [8]

$$V(x(t), \dot{x}(t), w(t)) = \frac{1}{2}\dot{x}(t)^T\dot{x}(t) + \frac{1}{2}x(t)^T Ax(t)$$
$$+ \frac{1}{2}\frac{a}{b}w(t)^2 + \frac{1}{2}K[w(t) + B^T x(t)]^2. \tag{33}$$

We can notice that $V(x, \dot{x}, w) > 0$ and $V(x, \dot{x}, w) \to \infty$ if $[x \ \dot{x} \ w] \to \infty$. From (33) and (32) and from elementary calculations we finally obtained

$$\dot{V}(x(t), \dot{x}(t), w(t)) = -b\{[\frac{a}{b} + K]w(t) + KB^T x(t)\}^2$$
$$\leqslant 0. \tag{34}$$

By LaSalle's theorem [6] the solutions of (32) asymptotically tends to the maximal invariant subset of E, where

$$E = \{(x, \dot{x}, w) : \dot{V} = 0\}. \tag{35}$$

From $\dot{V} = 0$ we have $s(t) = 0$, $t \geqslant 0$ (see (30) with $\alpha = \frac{a}{b} + K$ and $\beta = K$). The system (30) is observable (see remark 9), thus from $s(t) = 0$, $t \geqslant 0$ we have $x = 0$, $\dot{x} = 0$, $w = 0$ and (see (35)) finally it is easy to see that the largest invariant set contained in $E = \{0\}$ is the set $S = \{0\}$. We have proved the theorem. □

7. Numerical examples

Our computations were performed using MATLAB package. Let's consider the undamped system (23), (2) with $LC = 1$ and $n = 5$. Let $x_1(0) = 0.2$, $x_i(0) = 0$, $i = 2, 3, 4, 5$. In Fig. 2 output trajectory $y(t) = x_1(t)$ for $K = 0.0$ is shown (see (31)).

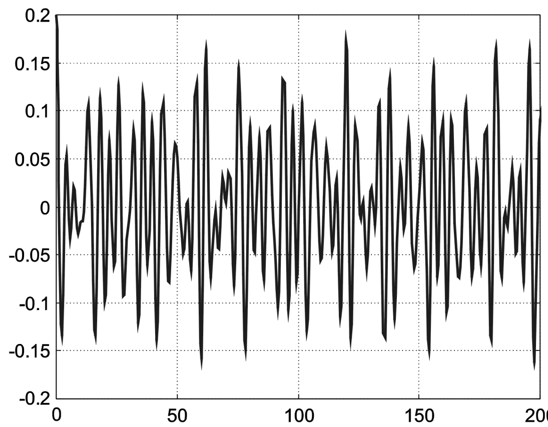

Fig. 2. Trajectory $y(t) = x_1(t)$ of undamped system (23)

Let $G_1 = G$, $G_2 = 0$ (see Fig. 1; stabilisation by inner resistance). In Fig. 3 an output trajectory $y(t) = x_1(t)$ for $G = 1$, $C = 1$ is shown.

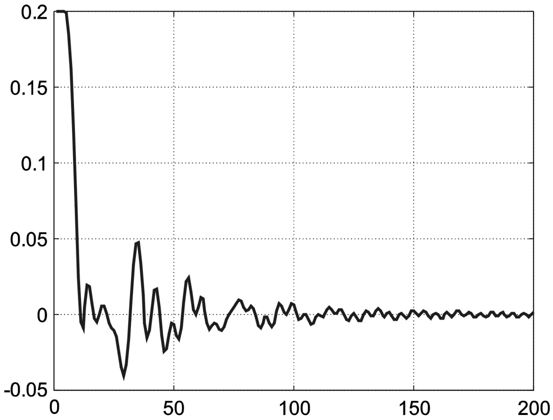

Fig. 3. Stabilisation by inner resistance with $G = 1$

Now we consider the stabilisation of the undamped system (23), (2) by the first range dynamic feedback (29). In Fig. 4 the trajectory $y(t) = x_1(t)$ of closed-loop system (32) for $K = 0.5$ and $a = 1$, $b = 1$, $w(0) = 0$ is shown (see (29)).

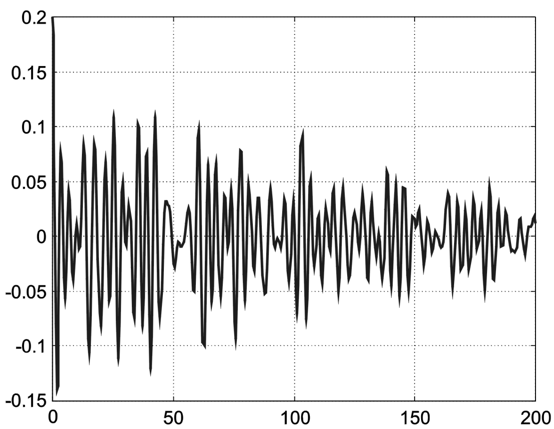

Fig. 4. Trajectory $y(t) = x_1(t)$ with feedback (29) for $K = 0.5$ and $a = 1$, $b = 1$

In Fig. 5 the trajectory $y(t) = x_1(t)$ of closed-loop system (32) for $K = 5$ and $a = 1$, $b = 1$, $w(0) = 0$ (see (29)) is shown.

The quality of stabilisation for $K = 5$ is higher then quality of stabilisation for $K = 0.5$ (see Fig. 4 and Fig. 5).

In Fig. 6 and 7 there is the root locus for $K > 0$ which shows that the closed-loop system given by (32) is asymptotically stable. In Fig. 6 and 7 $K > 0$ is equal to: 0, 0.0034, 0.0080, 0.0190, 0.0448, 0.1059, 0.2503, 0.5912, 1.3968, 2.3484, 3.3000 and ∞ respectively.

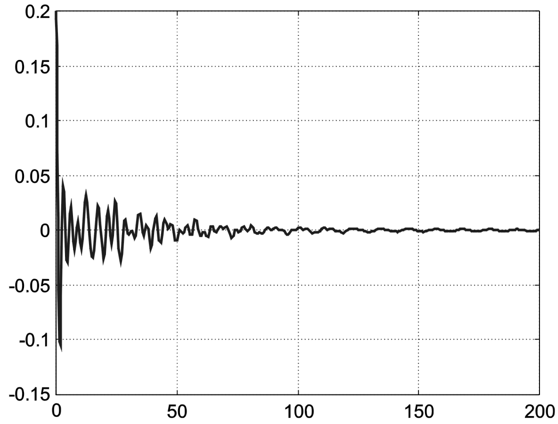

Fig. 5. Trajectory $y(t) = x_1(t)$ with feedback (29) for $K = 5$ and $a = 1$, $b = 1$

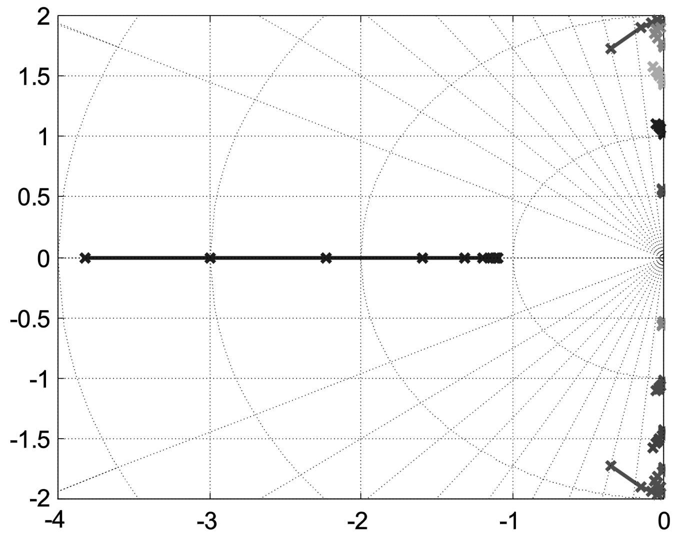

Fig. 6. Root locus for $K > 0$

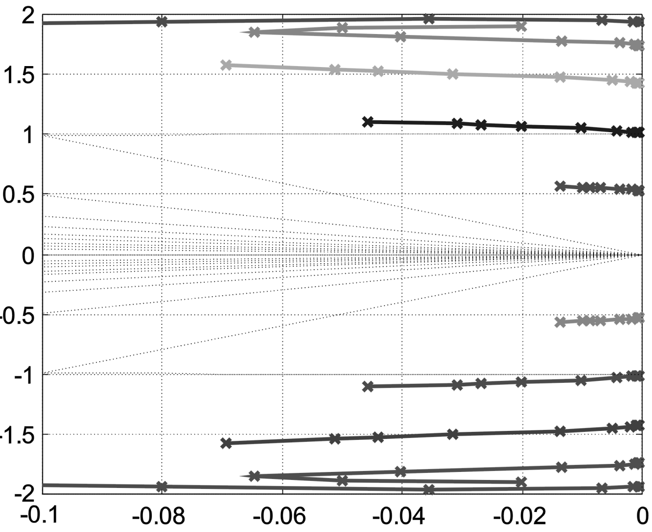

Fig. 7. Root locus for $K > 0$ (magnification)

Remark 10. The quality of stabilisation can be characterised by norm of matrix $e^{\tilde{A}t}$, where $\tilde{A}$ is the state matrix of closed loop system (32). □

Let $H(A) = \max_k \sqrt{\lambda_k(A^T A)}$ be the spectral norm of matrix A [3]. In Figs. 8 and 9 there is shown the spectral norm's $H(e^{\tilde{A}t})$ for $K = 0$ and $K = 5$ respectively, where $\tilde{A}$ is the state matrix of closed loop system (32). In this case $LC = 1$, $a = 1$. $b = 1$ and $n = 5$.

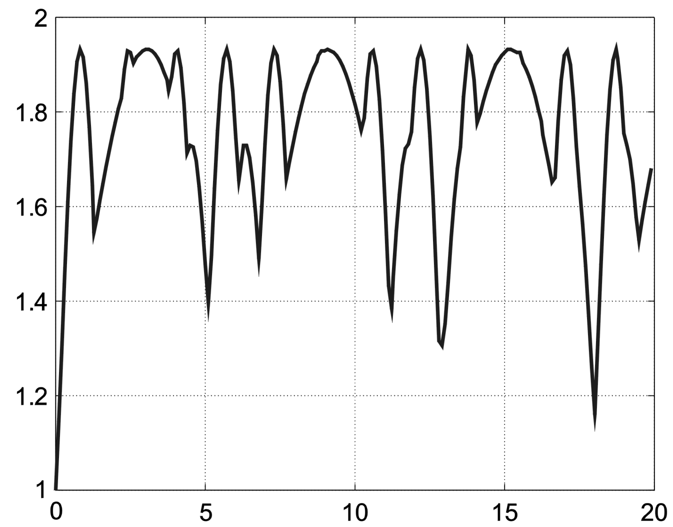

Fig. 8. Spectral norm's of matrix $e^{\tilde{A}t}$ for $K = 0$ and $t \in [0, 20]$

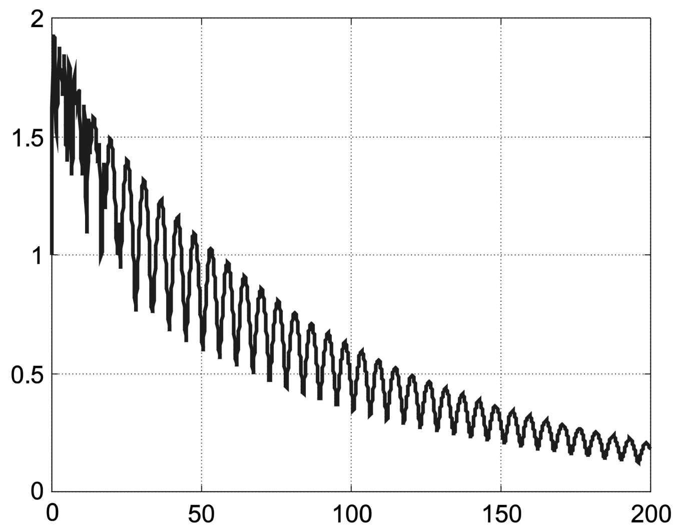

Fig. 9. Spectral norm's of matrix $e^{\tilde{A}t}$ for $K = 5$ and $t \in [0, 200]$

8. Remarks

In this paper, we considered the stabilisation problem of the LC-ladder system (1) with $D = 0$ (undamped system). We studied the cases: stabilisation by inner resistance (see *Remark* 6 and Theorem 1), by velocity feedback (see *Remark* 8) and stabilisation by dynamic linear feedback, in particularly stabilisation by first range dynamic feedback (see Theorem 2). To prove that the respectively system is globally asymptotically stable, we have used LaSalle's invariance principle [6]. In our stabilisation problem the observability of the system (1) with proper output played an essential role. Illustrative examples show the quality of stabilisation.

Numerical calculations were made using the MATLAB program.

Acknowledgements. Presented work was sponsored by KBN-AGH Contract No. 11 11 120 230.

References

[1] W. Mitkowski, *Stabilization of Dynamic Systems*, WNT, Warszawa, 1991, (in Polish).

[2] W. Mitkowski, "Dynamic feedback in LC ladder network", *Bull. Pol. Ac.: Tech.* 51(3), 173–180 (2003).

[3] A. Turowicz, *Theory of Matrix*, 5 ed., AGH, Kraków (1995), (in Polish).

[4] T. Kaczorek, *Matrix in Automatics and Electrotechnics*, WNT, Warszawa (1984), (in Polish).

[5] J. Klamka, *Controllability of Dynamic Systems*, PWN, Warszawa (1990), (in Polish).

[6] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, PWN, Warszawa (1966), (in Polish).

[7] T. Kobayashi, "Low gain adaptive stabilization of undamped second order systems", *Archives of Control Sciences* 11(1–2), 63–75 (2001).

[8] W. Mitkowski and P. Skruch, "Stabilization of second order systems by linear position feedback", *Proc. MMAS 2004*, (to be published).